Transport Functors and Connections on Gerbes, Part I [1]

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on joint work with Urs Schreiber

Abstract

Parallel transport in a fibre bundle with connection can be seen as an assignment of fibres to points and of morphisms between fibres to paths, together defining a functor. I describe a characterization of these functors resulting in a category of "transport functors" that is equivalent to the category of fibre bundles with connection.

Let M be a smooth manifold and let P be a principal G-bundle over M with connection ω.

- Each fibre P_x is a G-torsor, i.e. a smooth manifold with a smooth, free and transitive G-action from the right.
- The parallel transport along a curve $\gamma : x \longrightarrow y$ is a G-equivariant smooth map $\tau_\gamma : P_x \longrightarrow P_y$.

What are the abstract properties of these maps τ_γ?

1. For $\gamma_1 : x \longrightarrow y$ and $\gamma_2 : y \longrightarrow z$ composable paths, $\tau_{\gamma_2 \circ \gamma_1} = \tau_{\gamma_2} \circ \tau_{\gamma_1}$.
2. For id_x the constant path at a point x, $\tau_{\mathrm{id}_x} = \mathrm{id}_{P_x}$.

Observation. These are the axioms of a functor

$$\mathrm{tra}_{P,\omega} : \mathcal{P}_1(M) \longrightarrow G\text{-Tor},$$

where

[1] This is the first of two lectures given in August 2008 at the Topology Seminar of the University of California at Berkeley. It is based on the article "Parallel Transport and Functors" (`arxiv:0705.0452`).

- $\mathcal{P}_1(M)$ is the path groupoid of M: its objects are the points in M, and its morphisms are *thin homotopy classes* of paths in M.
- G-Tor is the category of G-torsors.

Here we have used the well-known fact that the parallel transport maps τ_γ only depend on the thin homotopy class of γ.
In order to make the definition of the path groupoid a bit more precise, let us denote by PM the set of paths in M: smooth maps $\gamma : [0,1] \longrightarrow M$ with sitting instants. On this set there is an equivalence relation $\sim$ called "thin homotopy" according to which two paths γ_1 and γ_2 are equivalent if there exists a smooth homotopy whose differential is at most rank 1. Then, the quotient $PM/_\sim$ is the set of morphisms of $\mathcal{P}_1(M)$.

Question. For which functors

$$F : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$$

exists a principal G-bundle P with connection ω such that $F = \mathrm{tra}_{P,\omega}$?

To approach this question, let us restrict first to trivial bundles.

A trivial principle G-bundle with connection over M is nothing but a 1-form $A \in \Omega^1(M, \mathfrak{g})$. Its functor

$$\mathrm{tra}_{G\times M,A} : \mathcal{P}_1(X) \longrightarrow G\text{-Tor} \tag{1}$$

has the particular form that every point $x \in M$ is mapped to G, as a G-torsor over itself. Hence, every parallel transport map $\tau_\gamma : G \longrightarrow G$ can be identified with a group element. The category G-Tor is thus much too large for this trivial situation.

Let us introduce a new category which is adapted to this particular situation. We call this category $\mathcal{B}G$: it has a single object, and its morphisms are the group elements of G. We have just identified the parallel transport of a trivial principal G-bundle with connection A with a functor

$$\mathrm{triv}_A : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G. \tag{2}$$

Remark. A concise way to specify the relation between the 1-form A and the functor triv_A is the path-ordered exponential

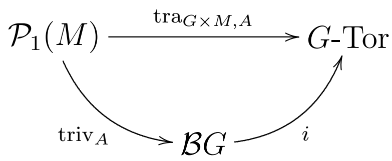
$$\mathrm{triv}_A(\gamma) = \mathcal{P}\exp\left(\int_0^1 \gamma^* A\right)$$

which stands for the solution of the differential equation one has to solve to compute the parallel transport.

The relation between the functors (1) and (2) is given by the functor

$$i : \mathcal{B}G \longrightarrow G\text{-Tor}$$

that sends the single object of $\mathcal{B}G$ to G, regarded as a G-torsor. Then, we have a commutative diagram:



New question. For which functors $F : \mathcal{P}_1(M) \longrightarrow \mathcal{B}G$ exists a 1-form $A \in \Omega^1(M, \mathfrak{g})$ such that $F = \mathrm{triv}_A$?

Observation: the adapted target category $\mathcal{B}G$ is a *Lie* category, in contrast to G-Tor. We can thus put a *smoothness condition* on the functor F.

Definition 1.

1. *A plot is a map $c : U \longrightarrow PM$ defined on a smooth manifold U, such that the composite*

$$U \times [0,1] \xrightarrow{c\times \mathrm{id}} PM \times [0,1] \xrightarrow{\mathrm{ev}} M \tag{3}$$

is a smooth map.

2. *A functor $F : \mathcal{P}_1(M) \longrightarrow \mathcal{B}G$ is called smooth, if for every plot c the composite*

$$U \xrightarrow{c} PM \xrightarrow{\mathrm{pr}} PM/_{\sim} \xrightarrow{F} G$$

is a smooth map.

Remark. The set $PM/_{\sim}$ can be regarded a *diffeological space*. The definition above is equivalent to saying that $F : PM/_{\sim} \longrightarrow G$ is a diffeological map.

Lemma 2.

1. *The functor $\mathrm{triv}_A : \mathcal{P}_1(M) \longrightarrow \mathcal{B}G$ associated to a 1-form A is smooth.*

2. *The assignment*

$$\Omega^1(M, \mathfrak{g}) \longrightarrow \mathrm{Funct}^\infty(\mathcal{P}_1(M), \mathcal{B}G) \quad : \quad A \longmapsto \mathrm{triv}_A$$

is a bijection.

Proof. Consider a map $c : U \longrightarrow PM$ so that the composite (3) is smooth. For $p \in U$, the value $\mathrm{triv}_A(c(u))$ is the solution of an ordinary differential equation which depends smoothly on u. Hence, also the solution is a smooth function in u. The bijection is shown by defining in inverse, i.e. a 1-form A out of a smooth functor F. Let $\gamma : \mathbb{R} \longrightarrow M$ be a smooth curve, and define $c : [0,1] \longrightarrow PM$ where $c(t)$ is the path $c(t)(\tau) := \gamma(t\tau)$. This makes (2) smooth, so that also $F_\gamma := F \circ c : [0,1] \longrightarrow G$ is a smooth map, and $F_\gamma(0) = 1$. Then define

$$A_{\gamma(0)}(\dot{\gamma}(0)) := -\left.\frac{\partial}{\partial t}\right|_0 F_\gamma \in \mathfrak{g}.$$

This is independent of the choice of γ, and yields a globally defined differential form. □

Lemma 2 solves the trivial part of the question we started with: a functor $F : \mathcal{P}_2(M) \longrightarrow G\text{-Tor}$ is the transport functor of a *trivial* principal G-bundle with connection, if and only if it is of the form

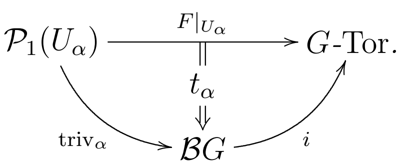
$$F = i \circ \mathrm{triv}$$

for a *smooth* functor $\mathrm{triv} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}G$.

Now we go on with the general situation. Since we know which functors stand for trivial principal G-bundles, we mimic the definition of a local trivialization.

Definition 3. *A* <u>*local trivialization*</u> *of a functor* $F : \mathcal{P}_1(M) \longrightarrow G\text{-Tor}$ *is*

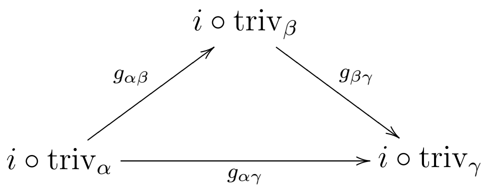
1. *a cover of* M *by open sets* U_α*,*
2. *functors* $\mathrm{triv}_\alpha : \mathcal{P}_1(U_\alpha) \longrightarrow \mathcal{B}G$ *and*
3. *natural equivalences*



The natural equivalences t_α induce natural equivalences

$$g_{\alpha\beta} := t_\beta \circ t_\alpha^{-1} : i \circ \mathrm{triv}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow i \circ \mathrm{triv}_\beta|_{U_\alpha \cap U_\beta},$$

the "transition transformations" of the transport functor. These satisfy the cocycle condition, i.e. $g_{\alpha\alpha} = \mathrm{id}$, and all diagrams



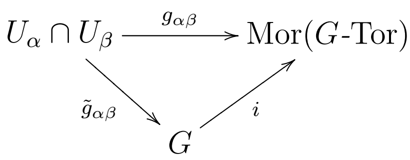
(4)

of natural equivalences are commutative. A collection $(\mathrm{triv}_\alpha, g_{\alpha\beta})$ satisfying (4) is called *descent data*.

Definition 4. *Descend data* $(\mathrm{triv}_\alpha, g_{\alpha\beta})$ *is called* <u>*smooth*</u>*, if the functors* triv_α *are smooth, and if there exist smooth maps*

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$$

such that all the diagrams



are commutative.

The concept of local trivializations and smooth descent data makes sense in a more general setup:

1. one can take any Lie groupoid Gr instead of $\mathcal{B}G$.
2. one can take any target category T instead of G-Tor.
3. one can take any functor $i : \mathrm{Gr} \longrightarrow T$.

Definition 5. *A* <u>*transport functor on M with* Gr*-structure*</u> *is a functor*

$$\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$$

which admits a local trivialization with smooth descent data.

Returning to our particular example, and answering the above question, we have

Theorem 6.

1. *The functor* $\mathrm{tra}_{P,\omega}$ *defined from a principal G-bundle P with connection ω, is a transport functor with $\mathcal{B}G$-structure.*

2. *Moreover, the functor*

$$G\text{-Bun}^{\nabla}(M) \longrightarrow \mathrm{Trans}^1_{\mathcal{B}G}(M, G\text{-Tor}) \quad : \quad (P,\omega) \longmapsto \mathrm{tra}_{P,\omega}$$

is surjective, full and faithful; in particular, an equivalence of categories.

Proof. We sketch only 1. here: Consider a local trivialization of the principal bundle, consisting of open sets U_α and smooth G-equivariant bundle maps $\phi_\alpha : U_\alpha \times G \longrightarrow P$. We have induced sections $s_\alpha : U_\alpha \longrightarrow P$ defined by $s_\alpha(x) := \phi(x, 1)$. The pullbacks

$$A_\alpha := s_\alpha^* \omega \in \Omega^1(U_\alpha, \mathfrak{g})$$

define by Lemma 2 the smooth functors $\mathrm{triv}_\alpha : \mathcal{P}_1(U_\alpha) \longrightarrow \mathcal{B}G$; these are the first ingredients of a local trivialization.
The natural equivalences $t_\alpha : \mathrm{tra}_{P,\omega}|_{U_\alpha} \longrightarrow i \circ \mathrm{triv}_\alpha$ are defined pointwise by

$$t_\alpha(x) := \phi_\alpha^{-1}|_{P_x} : P_x \longrightarrow G$$

for $x \in U_\alpha$, which is a morphism in G-Tor. To verify the commutativity of the naturality square

$$\begin{array}{ccc} P_x & \xrightarrow{t_\alpha(x)} & G \\ {\scriptstyle \tau_\gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{triv}_\alpha(\gamma)} \\ P_y & \xrightarrow[t_\alpha(y)]{} & G \end{array}$$

for a morphism $\gamma : x \longrightarrow y$ in $\mathcal{P}_1(U_\alpha)$ one checks that the way how $\mathrm{triv}_\alpha(\gamma)$ is computed corresponds to the way how τ_γ is computed usually. Finally, the associated transition transformations $g_{\alpha\beta}$ factor through the functor i by the ordinary transition functions $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$, which are smooth. □

Advantages. Describing fibre bundles with connection by transport functors has three main advantages:

1. More general classes of fibre bundles can be cooked up from the concept of a transport functor, for instance groupoid bundles with connection.

2. Transport functors on M induce tautologically functions on the loop space LM, since $LM \hookrightarrow PM$.

3. Transport functors have an evident categorification that leads to a systematical approach to connections on gerbes. This is the content of part II of this series.