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String Connections and Chern-Simons 2-Gerbes

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String structures on a principal $\mathrm{Spin}(n)$-bundle P over a smooth manifold M can be understood geometrically in two ways: (A) as lifts of the structure group of P from $\mathrm{Spin}(n)$ to a certain 3-connected cover, the *string group* [7], and (B) as lifts of the structure group of the looped bundle LP from $L\mathrm{Spin}(n)$ to its basic central extension [3]. I want to advertise a third way (C), which is equivalent to (A): string structures are trivializations of a certain geometrical object, namely a bundle 2-gerbe $\mathbb{CS}_P$ associated to P. In the following I want to outline the main results of my article [9] describing this approach.

The main advantage of my approach (C) is that the bundle 2-gerbe $\mathbb{CS}_P$ enjoys an explicit, smooth and finite-dimensional construction. This is in contrast to the approaches (A) and (B), which involve both non-smooth or infinite-dimensional smooth structures (the string group and loop spaces, respectively). I remark, however, that there is ongoing and promising research aiming at a finite-dimensional and smooth replacement for the string group in terms of certain generalized Lie 2-groups [6].

The bundle 2-gerbe $\mathbb{CS}_P$ is a certain *Chern-Simons bundle 2-gerbe* [1]. Let me give the idea of its construction. We start with a given principal $\mathrm{Spin}(n)$-bundle P over M. The 2-fold fibre product $P^{[2]} := P \times_M P$ comes with a canonical map $g : P^{[2]} \to \mathrm{Spin}(n)$ which expresses the fact that P trivializes canonically when pulled back to its own total space. Over $\mathrm{Spin}(n)$ one finds the *basic bundle gerbe* $\mathcal{G}$, whose Dixmier-Douady class is the generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z}) \cong \mathbb{Z}$. There exists a Lie-theoretic construction of $\mathcal{G}$ due to Gawędzki-Reis [2] and Meinrenken [4], finite-dimensional and smooth. The pullback of $\mathcal{G}$ along the map g is one part of the Chern-Simons 2-gerbe. The remaining ingredients are provided by a *multiplicative structure* on $\mathcal{G}$.

Like every bundle 2-gerbe, the Chern-Simons 2-gerbe has a characteristic class in $\mathrm{H}^4(M, \mathbb{Z})$. This class is

$$[\mathbb{CS}_P] = \tfrac{1}{2}p_1(P) \in \mathrm{H}^4(M, \mathbb{Z}),$$

the obstruction against string structures in the Stolz-Teichner approach (A). As a consequence, string structures on P exist if and only if $\mathbb{CS}_P$ admits trivializations. The situation is even better: there exists a canonical bijection between isomorphism classes of trivializations of $\mathbb{CS}_P$ and equivalence classes of string structures in the Stolz-Teichner approach (A). Summarizing, trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ are a geometrical, smooth and finite-dimensional way to describe string structures.

One can now lift the whole construction to a setup with connections. This benefits particularly from the fact that we have only involved smooth, finite-dimensional manifolds. We assume that the principal Spin(n)-bundle P comes equipped with a connection A. One can show that this connection defines a canonical connection ∇_A on $\mathbb{CS}_P$. Let me just mention that part of this connection is a 3-form on P, namely the Chern-Simons 3-form $TP(A)$. Now we can look at trivializations of $\mathbb{CS}_P$ that respect the connection ∇_A in a certain way. This actually means to equip a trivialization with additional structure, that we call *string connection*. In my article [9] I show that

- To every string structure and every connection A on P there exists a string connection.
- The set of possible choices forms a contractible space.

The collection of a string structure and a string connection is called a *geometric string structure*. This notion of a geometric string structure has a number of interesting properties, which I want to outline in the following.

- Geometric string structures on (P, A) form a 2-groupoid, which is a module over the 2-groupoid of bundle gerbes with connection over M.
- On isomorphism classes, one obtains a free and transitive action of the differential cohomology $\hat{\mathrm{H}}^3(M, \mathbb{Z})$ on the set of isomorphism classes of geometric string structures is induced.
- Associated to every geometric string structure is a 3-form $H \in \Omega^3(M)$ whose pullback to P differs from the Chern-Simons 3-form $TP(A)$ by a closed 3-form with integral periods.

I remark that the notion of a geometric string structure in my approach (C) coincides with the original definition given by Stolz and Teichner [7] in the sense that both trivialize a certain Chern-Simons theory.

Another interesting link is to Redden's thesis [5], in which he constructs another 3-form $H_{g,A}$ associated to a string structure, a connection A on P, and a Riemannian metric g on M. One would like to have string connection associated to g and A, such that the two 3-forms coincide, $H = H_{g,A}$. During the workshop, Redden and I could at least show that such a string connection always exists.

Let me finally outline how my approach (C) to string structures relates to approach (B), namely to lifts of the structure group of LP from $L\mathrm{Spin}(n)$ to its basic central extension. For this purpose we look at the *transgression* of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ to the free loop space LM. This is a bundle gerbe $\mathscr{T}_{\mathbb{CS}_P}$ over LM that one can explicitly construct from the given bundle 2-gerbe. On the level of characteristic classes, the construction covers the transgression homomorphism

$$\mathrm{H}^4(M, \mathbb{Z}) \to \mathrm{H}^3(LM, \mathbb{Z}).$$

What has this bundle gerbe $\mathscr{T}_{\mathbb{CS}_P}$ over LM to do with string structures? We use a result from [8] showing that the transgression of the basic bundle gerbe $\mathcal{G}$ defines a principal U(1)-bundle over LM, which underlies the basic central extension

$$1 \to \mathrm{U}(1) \to \widehat{L\mathrm{Spin}(n)} \to L\mathrm{Spin}(n) \to 1.$$

This fact makes the relation between $\mathbb{CS}_P$, which we have constructed using the basic bundle gerbe $\mathcal{G}$, and the string structures in the approach (B). More precisely, the bundle gerbe $\mathscr{T}_{\mathbb{CS}_P}$ *is* the lifting bundle gerbe associated to the problem of lifting the structure group of LP along the above central extension. As a consequence, string structures in the sense of trivializations of $\mathbb{CS}_P$ transgress to string structures in the sense of McLaughlin.

References

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