Multiplicative Gerbes and Chern-Simons Theory

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1 Gerbes and Lie Groups

$\rightsquigarrow$ Sections 1 and 2 are based on [SW09].

n-Gerbes $\approx$ geometrical objects over smooth manifolds, such that

$$\left\{ \begin{array}{c} n\text{-gerbes over } M\text{, up} \\ \text{to isomorphism} \end{array} \right\} \cong \mathrm{H}^{n+2}(M,\mathbb{Z})$$

Various versions of n-gerbes possible, my favorite ones are:

- (-1)-gerbe = smooth map $M \longrightarrow S^1$
- 0-gerbe = principal S^1-bundle over M
- 1-gerbe = bundle gerbe
- 2-gerbe = bundle 2-gerbe

  $\vdots$

Bundle gerbe (Murray '95):

1. Surjective submersion $\pi : Y \longrightarrow M$
2. principal S^1-bundle L over $Y^{[2]} = Y \times_M Y$

3. bundle isomorphism $\mu : \pi_{12}^* L \otimes \pi_{23}^* L \longrightarrow \pi_{13}^* L$ over $Y^{[3]}$, associative.

Class in $\mathrm{H}^3(M, \mathbb{Z})$ associated to bundle gerbe $\mathcal{G}$ called *Dixmier-Douady* class, denoted $\mathrm{DD}(\mathcal{G})$.

Gerbes particularly interesting when $M = G$ a compact, simple and simply-connected Lie group:

$$\mathrm{H}^3(G, \mathbb{Z}) = \mathbb{Z} \quad \Longrightarrow \quad \text{canonical } \mathbb{Z}\text{-family of isomorphism classes of bundle gerbes}$$

Even better: canonical $\mathbb{Z}$-family of bundle gerbes $\mathcal{G}_k$ over G. Recall Lie-theoretical construction of $\mathcal{G}_k$ (Gawędzki-Reis [GR03], Meinrenken [Mei02]):

- $Y := \bigsqcup U_\alpha$ disjoint union of open sets of a cover of G, labelled by vertices α of a Weyl alcove $\mathfrak{A} \subset \mathfrak{g}^*$:
$$U_\alpha = q^{-1}(\mathfrak{A} \setminus f_\alpha)$$
where $q : G \longrightarrow \mathfrak{A}$ picks the element $q(g) \in \mathfrak{A}$ that corresponds to the conjugacy class of g, and f_α is the closed face of $\mathfrak{A}$ opposite of α.

- Any intersection $U_{\alpha_1} \cap U_{\alpha_2}$ can be identified with the coadjoint orbit $\mathcal{O}_{\alpha_2 - \alpha_1} \subset \mathfrak{g}^*$.

  For $G = \mathrm{SU}(n), \mathrm{Sp}(n)$, $\mathcal{O}_{\mu_2 - \mu_1}$ is integrable: canonical "prequantum" principal S^1-bundle $\mathcal{L}_{\alpha_2 - \alpha_1}$ over $\mathcal{O}_{\alpha_2 - \alpha_1}$. Union of these define L.

- Isomorphism μ obtained by canonical identification
$$\mathcal{L}_{\alpha_3 - \alpha_1} = \mathcal{L}_{\alpha_2 - \alpha_1 + \alpha_3 - \alpha_2} \cong \mathcal{L}_{\alpha_2 - \alpha_1} \otimes \mathcal{L}_{\alpha_3 - \alpha_2}.$$

2 Connections

n-gerbes *with connection* are supposed to realize *differential* cohomology:

$$\left\{ \begin{array}{c} n\text{-gerbes over } M \text{ with connection,} \\ \text{up to connection-preserving} \\ \text{isomorphisms} \end{array} \right\} \cong \hat{\mathrm{H}}^{n+2}(M, \mathbb{Z})$$

- connection on a (-1) gerbe = no information
- connection on a 0-gerbe = connection on the principal S^1-bundle
- connection on a bundle gerbe (Y, π, L, μ):

1. a connection on the S^1-bundle L
2. a 2-form $B \in \Omega^2(Y)$

such that μ is connection-preserving and

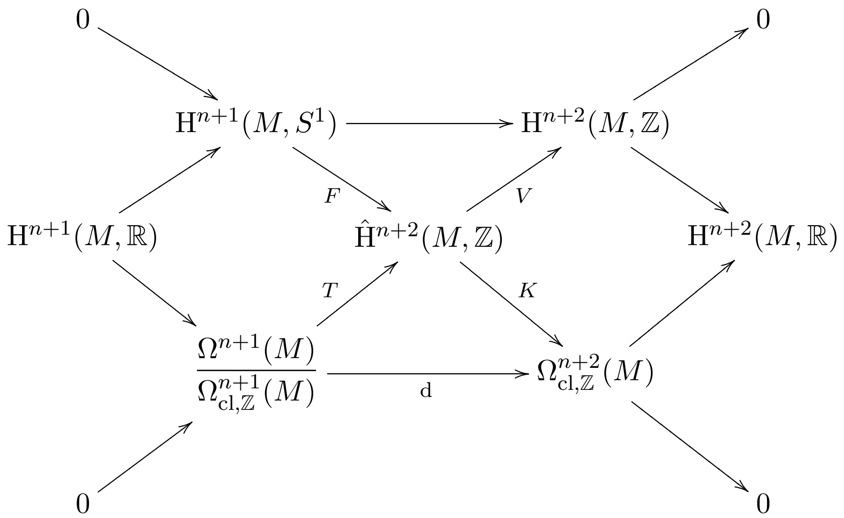
$$\pi_2^* B - \pi_1^* B = \mathrm{curv}(L).$$

Two important constructions for bundle gerbes with connection:

1. Curvature = unique 3-form $H \in \Omega^3(M)$ such that $\pi^* H = \mathrm{d}B$.
2. Trivial bundle gerbe with connection $\mathcal{I}_\rho$ associated to $\rho \in \Omega^2(M)$:

   $Y = M$, $\pi = \mathrm{id}$, $L = M \times S^1$ equipped with the trivial flat connection, $\mu = \mathrm{id}$ and $B := \rho$.

Recall differential cohomology. Universal characterization by *character diagram* (Simons-Sullivan [SS]):



All subdiagrams are supposed to be commutative, and the two diagonal short sequences are supposed to be exact.

Upon realizing $\hat{\mathrm{H}}^3(M, \mathbb{Z})$ by isomorphism classes of bundle gerbes over M with connection:

- V is "forgetting the connection"
- K is the curvature

- T produces the trivial gerbe $\mathcal{I}_\rho$ (up to isomorphism)
- F produces a bundle gerbe with flat connection (not needed in the following)

Use character diagram to define the holonomy of an n-gerbe $\mathcal{G}$ with connection:

1. $\phi : \Sigma \longrightarrow M$ smooth map with Σ $(n+1)$-dimensional, closed, oriented
2. Pullback $\phi^*\mathcal{G}$ has vanishing class in $\mathrm{H}^{n+2}(\Sigma, \mathbb{Z})$. Exactness:
$$\phi^*\mathcal{G} \cong \mathcal{I}_\rho$$
for some $\rho \in \Omega^{n+1}(\Sigma)$.
3. Holonomy
$$\mathrm{Hol}_{\mathcal{G}}(\phi) := \int_\Sigma \rho \quad \in \mathbb{R}/\mathbb{Z}$$
well-defined, since differences of ρ's lie in $\Omega^{n+1}_{\mathrm{cl},\mathbb{Z}}(\Sigma)$.

Canonical bundle gerbe $\mathcal{G}_k$ over Lie group G has canonical connection of curvature

$$H_k := k \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G),$$

with θ left-invariant Maurer-Cartan form on G, and $\langle -, - \rangle$ normalized such that H_1 represents $1 \in \mathbb{Z} = \mathrm{H}^3(G, \mathbb{Z})$.

3 Multiplicative Gerbes

$\rightsquigarrow$ Sections 3 and 4 are based on [Wala].

Want compatibility of a gerbe $\mathcal{G}$ over G with the group structure. Possible:

1. Jandl gerbe: $i^*\mathcal{G} \cong \mathcal{G}^*$ with $i : G \longrightarrow G$ the inversion (see [SSW07])
2. Equivariant gerbe: $c^*\mathcal{G} \cong p_2^*\mathcal{G}$ for $c : G \times G \longrightarrow G$ conjugation action
3. Multiplicative Gerbe: isomorphism
$$\mathcal{M} : p_1^*\mathcal{G} \otimes p_2^*\mathcal{G} \longrightarrow m^*\mathcal{G}$$
with $m, p_1, p_2 : G \times G \longrightarrow G$ multiplication and the two projections.

Remarks:

- suppress higher coherence data and axioms in this talk
- 3. contains 1. and 2. as particular cases
- Canonical gerbes $\mathcal{G}^k$ are multiplicative

Classification of multiplicative gerbes:

$$\begin{array}{ccc} (\mathcal{G},\mathcal{M}) \in \left\{ \begin{array}{c} \text{Multiplicative} \\ \text{bundle gerbes} \\ \text{over } G\text{, up to iso} \end{array} \right\} & \xrightarrow[\text{Carey et al.[CJM}^+\text{05]}]{\cong} & \mathrm{H}^4(BG,\mathbb{Z}) \\ \downarrow \qquad\qquad \downarrow & & \downarrow \text{Transgression} \\ \mathcal{G} \quad \in \quad \left\{ \begin{array}{c} \text{Gerbes over} \\ G\text{, up to iso} \end{array} \right\} & \xrightarrow{\cong} & \mathrm{H}^3(G,\mathbb{Z}) \end{array}$$

Connections on multiplicative gerbes difficult:

1. Naive definition: connection on $\mathcal{G}$ and $\mathcal{M}$ connection-preserving
2. Then, curvature H of $\mathcal{G}$ satisfies $\Delta H := m^*H - p_1^*H + p_2^*H = 0$.
3. Problem: curvature H_k of canonical bundle gerbe $\mathcal{G}_k$ satisfies only $\Delta H = \mathrm{d}\rho_k$, with
$$\rho_k = \frac{k}{2}\left\langle p_1^*\theta \wedge p_2^*\bar{\theta}\right\rangle \in \Omega^2(G \times G);$$
$\implies$ $\mathcal{G}_k$ would not be multiplicative.

Better definition includes the 2-form:

Definition 1. *A multiplicative bundle gerbe with connection over G is a triple $(\mathcal{G},\rho,\mathcal{M})$, where*

- *$\mathcal{G}$ is a bundle gerbe with connection over G; denote by H its curvature*
- *ρ is a 2-form on $G \times G$ such that $\Delta H = \mathrm{d}\rho$ and $\Delta\rho = 0$.*
- *$\mathcal{M}$ is a connection-preserving isomorphism $p_1^*\mathcal{G} \otimes p_2^*\mathcal{G} \longrightarrow m^*\mathcal{G} \otimes \mathcal{I}_\rho$*

This definition achieves its primary goal:

Theorem 2 ([Wala]). *The canonical bundle gerbes $\mathcal{G}_k$ with their canonical connection of curvature H_k are multiplicative with 2-form ρ_k in a unique way.*

On simple, compact but non-simply connected Lie groups G, the forms H_k and ρ_k still make sense.

Theorem 3 (with K. Gawędzki [GW09]). *G compact and simple.*

(a) The values $k \in \mathbb{Z}$ for which gerbes $\mathcal{G}$ with connection of curvature H_k over G exist, and for which isomorphisms $\mathcal{M}$ making $(\mathcal{G}, \rho_k, \mathcal{M})$ multiplicative exist, are given by Table 1 below.

(b) If they exist, multiplicative structures on gerbes over G are unique.

$\tilde{G}$	Center	Z	$\mathcal{G}$ exists	$\mathcal{M}$ exists
$\mathrm{SU}(r)$	$\mathbb{Z}_r$	$Z = \mathbb{Z}_N$ with $N \mid r$	$2N \mid kr(r-1)$	$2N^2 \mid kr(r-1)$
$\mathrm{Spin}(2r+1)$	$\mathbb{Z}_2$	$Z = \mathbb{Z}_2$	–	$2 \mid k$
$\mathrm{Spin}(4r+2)$	$\mathbb{Z}_4$	$Z = \mathbb{Z}_2$	–	$2 \mid k$
		$Z = \mathbb{Z}_4$	$2 \mid k$	$8 \mid k$
$\mathrm{Spin}(4r)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$Z = \mathbb{Z}_2 \times \{0\}$	$2 \mid kr$	$4 \mid kr$
		$Z = \{0\} \times \mathbb{Z}_2$	–	$2 \mid k$
		$Z = \{(0,0),(1,1)\}$	$2 \mid kr$	$4 \mid kr$
		$Z = \mathbb{Z}_2 \times \mathbb{Z}_2$	$2 \mid kr$	$2 \mid k$ and $4 \mid kr$
$\mathrm{Sp}(2r)$	$\mathbb{Z}_2$	$Z = \mathbb{Z}_2$	$2 \mid kr$	$4 \mid kr$
E_6	$\mathbb{Z}_3$	$Z = \mathbb{Z}_3$	–	$3 \mid k$
E_7	$\mathbb{Z}_2$	$Z = \mathbb{Z}_2$	$2 \mid k$	$4 \mid k$

Table 1: The compact simple Lie group G is written as the quotient $\tilde{G}/Z$ of its universal covering group by a subgroup Z of the center of $\tilde{G}$. The table lists all possible covering groups $\tilde{G}$ and subgroups Z.

Multiplicative bundle gerbes with connection can be applied to:

- Central extensions of loop groups
- Symmetric bi-branes
- Chern-Simons theory (see next section)

} see [Wala]

- String structures and string connections (see [Walb])

4 The Chern-Simons 2-Gerbe

Classically, a Chern-Simons theory is defined by:

1. a simply-connected gauge group G with metric $\langle -, - \rangle$
2. a level $k \in \mathbb{Z}$

A field for (G, k) is a compact closed 3-manifold M with a principal G-bundle P with connection A. Associates to a field (M, P, A) is the Feynman amplitude

$$\mathcal{A}_{G,k}(M, P, A) := k \int_M s^* CS(A) \in \mathbb{R}/\mathbb{Z},$$

where

- $s : M \longrightarrow P$ is a section (every principal bundle with simply-connected structure group is trivializable over 3-manifolds)
- $CS(A) := \langle A \wedge \mathrm{d}A \rangle + \frac{2}{3} \langle A \wedge A \wedge A \rangle \in \Omega^3(P)$ is the Chern-Simons 3-form

What is a Chern-Simons theory for a general gauge group (where no section s may exist)?

Main idea [CJM+05]: realize the Feynman amplitude as the holonomy of a 2-gerbe with connection, the "Chern-Simons 2-gerbe".

To construct this 2-gerbe with connection, one needs:

1. a principal G-bundle P over some smooth manifold M with connection A,
2. a level $k \in \mathbb{Z}$ and
3. a multiplicative bundle gerbe $\mathcal{G}$ with connection over G of curvature H_k and with 2-form ρ_k.

Describe construction of the 2-gerbe $\mathbb{CS}_P(\mathcal{G})$:

1. Surjective submersion: $\pi : P \longrightarrow M$.

2. 3-form $C := kCS(A) \in \Omega^3(P)$.

3. Over $P^{[2]} = P \times_M P$, need a bundle gerbe $\mathcal{P}$ with connection. Take

$$\mathcal{P} := g^*\mathcal{G} \otimes \mathcal{I}_\omega$$

with $g : P^{[2]} \longrightarrow G$ given by $p_1.g(p_1, p_2) = p_2$ and $\omega := k \langle \pi_1^* A \wedge g^*\theta \rangle \in \Omega^2(P^{[2]})$. The "correction" by ω is necessary to achieve identity

$$\pi_2^* C - \pi_1^* C = \mathrm{curv}(\mathcal{P}).$$

4. Over $P^{[3]}$, need connection-preserving isomorphism

$$\mathcal{N} : \pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P} \longrightarrow \pi_{13}^*\mathcal{P}.$$

Take $\mathcal{N} := g_2^*\mathcal{M}$ with $g_2 : P^{[3]} \longrightarrow G \times G : (p_1, p_2, p_3) \longmapsto (g(p_1, p_2), g(p_2, p_3))$. Connection-preserving because $\mathcal{M}$ is connection-preserving and

$$\pi_{12}^*\omega + \omega_{23}^*\omega = \pi_{13}^*\omega + g_2^*\rho.$$

Again, higher coherence issues are suppressed.

Definition 4. *A Chern-Simons theory for a Lie group G is given by a level $k \in \mathbb{Z}$ and a multiplicative bundle gerbe $\mathcal{G}$ with connection over G of curvature H_k and with 2-form ρ. The Feynman amplitude is*

$$\mathcal{A}_{\mathcal{G}}(M, P, A) := \mathrm{Hol}_{\mathbb{CS}_P(\mathcal{G})}(M).$$

Remark:

1. in particular, a Chern-Simons theory has a class $\tau \in \mathrm{H}^4(BG, \mathbb{Z})$.
2. if P admits a section, $\mathrm{Hol}_{\mathbb{CS}_P(\mathcal{G})}(M) = k \displaystyle\int_M s^*CS(A)$.

Corollary 5. *For a compact and simple Lie group G, Table 1 lists all possible levels for which Chern-Simons theories exist.*

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