String structures and supersymmetric sigma models

Konrad Waldorf
Universität Greifswald

“Higher structures in string theory and quantum field theory”
Erwin-Schrödinger-Institut für Mathematische Physik

December 2015
1.) Supersymmetric sigma models and Pfaffian line bundles

2.) Spin structures on loop spaces

3.) String structures and string connections
A 2-dimensional **supersymmetric sigma model** consists of the following structure:

- the target space, a Riemannian manifold $M$.
- the world sheet, a Riemann surface $\Sigma$ with a spin structure $S$.

The fields are parameterized by world sheet embeddings

$$\phi \in M^\Sigma := C^\infty(\Sigma, M);$$

for each $\phi$ we have an associated Hilbert space of spinors,

$$\psi \in V_\phi := L^2(\Sigma, S \otimes \phi^* TM).$$

The action functional is

$$S(\phi, \psi) := \int_{\Sigma} \left\{ ||d\phi||^2 + \langle \psi, D_\phi \psi \rangle \right\} \, d\text{vol}_{\Sigma}.$$
A particular problem is to give rigorous sense to the fermionic path integral

\[ A^\text{fer}(\phi) = \int_{\psi \in V_\phi} \exp \left( \int_{\Sigma} \langle \psi, \mathcal{D}_\phi \psi \rangle \, d\text{vol}_\Sigma \right) \, D\psi \]

which suffers from the absence of an appropriate measure.

Well-known solution: associate to each \( \phi \) a complex line \( P_\phi \) and identify \( A^\text{fer}(\phi) \) as a well-defined element in \( P_\phi \).

Varying \( \phi \) over \( M^\Sigma := C^\infty(\Sigma, M) \), the complex lines \( P_\phi \) form a smooth line bundle \( \text{Pfaff}(\mathcal{D}) \) over \( M^\Sigma \), and the elements \( A^\text{fer}(\phi) \) form a smooth section \( A^\text{fer} \in \Gamma(M^\Sigma, \text{Pfaff}(\mathcal{D})) \).
The space $M^\Sigma = C^\infty(\Sigma, M)$ of bosonic fields parameterizes a family of $\mathbb{Z}_2$-graded Hilbert spaces

$$\mathcal{H}_\phi := L^2(\Sigma, \mathbb{S} \otimes_{\mathbb{R}} \phi^* TM).$$

On every Hilbert space $\mathcal{H}_\phi$ we have the **Dirac operator** $D$ on $\mathbb{S}$ twisted by the Levi-Civita connection $\phi^* \nabla$ on $M$, and additionally twisted by a natural quaternionic structure $J$ on $\mathbb{S}$,

$$\mathcal{D}_\phi := J \circ (D \otimes \phi^* \nabla).$$

Thus, $\mathcal{D}_\phi$ is an even, anti-self-adjoint operator on $\mathcal{H}_\phi$. 
We regard the even, anti-self-adjoint operator $\mathcal{D}_\phi$ as a skew-symmetric bilinear form 

$$( - , \mathcal{D}_\phi - ) := \int_\Sigma \langle - , \mathcal{D}_\phi - \rangle \, \text{d} \text{vol}_\Sigma.$$ 

We introduce a spectral cut $\mu > 0$ for $\mathcal{D}_\phi$, and obtain an $2k$-dimensional vector space $\mathcal{H}_\phi^\mu,^+$, equipped with the skew form 

$$( - , \mathcal{D}_\phi - ) \in \Lambda^2(\mathcal{H}_\phi^\mu,^+)^*.$$ 

It defines an element 

$$p\text{faff}_\phi^\mu := \frac{1}{k!} ( - , \mathcal{D}_\phi - )^k \in \Lambda^{2k}(\mathcal{H}_\phi^\mu,^+)^* =: \det \mathcal{H}_\phi^\mu,^+.$$
The **Berezin integral** is defined for any finite-dimensional vector space $V$:

$$
\int_V : \Lambda^p V^* \longrightarrow \det V^* : \alpha \mapsto \begin{cases} 
\alpha & \text{if } p = \dim V \\
0 & \text{else}
\end{cases}
$$

If $\dim V = 2k$ and $\alpha \in \Lambda^2 V^*$, then

$$
\int_V \exp(\alpha) = \frac{1}{k!} \alpha^{\wedge k}.
$$

We apply this to $V = \mathcal{H}_{\phi}^{\mu,+}$ and $\alpha = (-, \hat{\mathcal{D}}_\phi -)$. Then we have rigorously interpreted

$$
\int_{\mathcal{H}_{\phi}^{\mu,+}} \exp \left( \int_{\Sigma} \langle -, \hat{\mathcal{D}}_\phi - \rangle \, d\text{vol}_\Sigma \right) = \text{pfaff}_\phi^{\mu} \in \det \mathcal{H}_{\phi}^{\mu,+}.
$$
It remains to get rid of the spectral cut $\mu$. We work over the open set

$$U_{\mu} := \{ \phi \in B \mid \mu \notin \text{spec}(\mathcal{D}_\phi) \}.$$  

$\mathcal{H}^{\mu, +}_\phi$ are fibres of a smooth, finite-dimensional vector bundle $\mathcal{H}^{\mu, +}$. $\text{pfaff}^{\mu}_\phi$ are the values of a smooth section $\text{pfaff}^{\mu}$ of $\det(\mathcal{H}^{\mu, +})$.

The open sets $U_{\mu}$ cover $M^\Sigma$. One can glue the determinant line bundles $\det(\mathcal{H}^{\mu, +})$ in two different ways:

1.) one obtains the usual determinant line bundle $\det \mathcal{D}$

2.) one obtains a line bundle $Pfaff(\mathcal{D})$, the Pfaffian line bundle.

The sections $\text{pfaff}^{\mu}$ glue to a global section $\text{pfaff}$ of $Pfaff(\mathcal{D})$. 
Summarizing, the fermionic path integral is rigorously defined by

\[ A^{fer}(\phi) := pfaff(\phi), \]

forming a smooth section \( A^{fer} \in \Gamma(M^\Sigma, Pfaff(\mathcal{D})) \).

Thus, the integrand for the full path integral,

\[ A(\phi) = \exp \left( \int_{\Sigma} \|d\phi\|^2 \cdot dvol_{\Sigma} \right) \cdot A^{fer}(\phi) \]

is a smooth section of \( Pfaff(\mathcal{D}) \).

It is not a function \( A : M^\Sigma \rightarrow \mathbb{C} \). This situation is called an anomaly ("global", "fermionic",...) . Our mission is to cancel this anomaly, for instance by providing a trivialization of \( Pfaff(\mathcal{D}) \).
1.) Supersymmetric sigma models and Pfaffian line bundles

2.) Spin structures on loop spaces

3.) String structures and string connections
We want to trivialize the line bundle $Pfaff(\mathcal{D})$ over $M^\Sigma = C^\infty(\Sigma, M)$.

**Theorem (Freed ’03)**

*If $M$ is equipped with a spin structure, then*

$$c_1(Pfaff(\mathcal{D})) = \int_\Sigma ev^*(\frac{1}{2}p_1(M))$$

*where $ev : M^\Sigma \times \Sigma \rightarrow M$ is the evaluation map, and $\frac{1}{2}p_1(M) \in H^4(M, \mathbb{Z})$ is the first fractional Pontryagin class of $M$.***

In particular, $Pfaff(\mathcal{D})$ is trivializable if $\frac{1}{2}p_1(M) = 0$. Spin manifolds that satisfy this condition are called **string manifolds**.

But we need more: we need a trivialization of $Pfaff(\mathcal{D})$. 
For the 2-torus $\Sigma = S^1 \times S^1$, integration factors through the free loop space $LM := C^\infty(S^1, M)$:

\[
\begin{array}{cccc}
\mathbb{H}^4(M, \mathbb{Z}) & \xrightarrow{\int_{S^1} ev^*} & \mathbb{H}^3(LM, \mathbb{Z}) & \xrightarrow{\int_{S^1} ev^*} & \mathbb{H}^2(M^\Sigma, \mathbb{Z}) \\
\frac{1}{2} p_1(M) & \rightarrow & \lambda & \rightarrow & c_1(Pfaff(\mathcal{D}))
\end{array}
\]

The intermediate step $\lambda \in \mathbb{H}^3(LM, \mathbb{Z})$ is an analog of the 3rd integral Stiefel-Whitney class for the loop space.

We see that $Pfaff(\mathcal{D})$ is trivializable if $\lambda = 0$. 
Let $FM$ be the frame bundle of $M$, with the structure group reduced to $\text{Spin}(n)$.

**Theorem (Killingback '87; McLaughlin '92)**

$\lambda$ vanishes if and only if the structure group of $LFM$ can be reduced to the universal loop group extension

$$1 \rightarrow U(1) \rightarrow \widehat{L\text{Spin}}(n) \rightarrow L\text{Spin}(n) \rightarrow 1.$$

Such a reduction is called **spin structure** on $LM$.

Killingback’s idea: a spin structure on $LM$ should give a trivialization of $\text{Pfaff}(\emptyset)$. However, this has never been confirmed.
The relation between the class $\lambda \in \mathbb{H}^3(LM, \mathbb{Z})$ and spin structures on $LM$ can be understood via the spin lifting gerbe. The spin lifting gerbe is a bundle gerbe over $LM$ with Dixmier-Douady class $\lambda$:

$$S_{LM} = \left\{ \begin{array}{c}
\mathcal{L} \rightarrow \widetilde{LSpin}(n) \\
\downarrow \\
\widetilde{LSpin}(n) \leftarrow \text{LFM}[2] \rightarrow \text{LFM} \\
\downarrow \\
LM
\end{array} \right\}$$

**Theorem (Murray '95)**

*Trivializations of $S_{LM}$ are in 1:1 correspondence with reductions, i.e. with spin structures on $LM$.***
1.) Supersymmetric sigma models and Pfaffian line bundles

2.) Spin structures on loop spaces

3.) String structures and string connections
We return to the original insight that $Pfaff(\mathcal{D})$ is trivializable if and only if $M$ is a string manifold, i.e. $\frac{1}{2}p_1(M) \in H^4(M, \mathbb{Z})$ vanishes.

Nowadays we have a nice higher-geometric structure which is classified by $H^4(M, \mathbb{Z})$: bundle 2-gerbes.

For the class $\frac{1}{2}p_1(M)$ there is a particularly nice bundle 2-gerbe: the Chern-Simons bundle 2-gerbe.

\[ CS_M = \left\{ \begin{array}{c}
 g^*G_{bas} \xrightarrow{\quad} G_{bas} \\
 \downarrow \quad \quad \quad \downarrow \\
 FM \xleftarrow{\quad} FM^{[2]} \quad \rightarrow Spin(n) \\
 \downarrow \quad \quad \quad \downarrow \\
 M
\end{array} \right. \]

(Carey-Johnson-Murray-Stevenson-Wang '05)
A trivialization of the Chern-Simons bundle 2-gerbe $CS_M$ consists of a bundle gerbe $S$ over $FM$ whose restriction to each fibre is $G_{bas}$.

Theorem (Stevenson '04)

A trivialization of $CS_M$ exists if and only if $\frac{1}{2}p_1(M) = 0$.

We call trivializations of the Chern-Simons 2-gerbe string structures.

Thus, we have the following implications:

$M$ admits string structures $\iff M$ is string $\implies Pfaff(\emptyset)$ is trivializable
The integration of cohomology classes

\[ H^4(M, \mathbb{Z}) \rightarrow H^3(LM, \mathbb{Z}) , \quad \frac{1}{2} p_1(M) \mapsto \lambda \]

\[ H^4(M, \mathbb{Z}) \rightarrow H^2(M^\Sigma, \mathbb{Z}) , \quad \frac{1}{2} p_1(M) \mapsto c_1(P) \]

lift to functors defined on the (homotopy) category of bundle 2-gerbes with connections:

\[ \mathcal{T}_{S_1} : h_1(2-\text{Grb}^\nabla(M)) \rightarrow h_1\text{Grb}(LM) \]

\[ \mathcal{T}_{\Sigma} : h_1(2-\text{Grb}^\nabla(M)) \rightarrow \text{LineBun}(M^\Sigma) \]

These functors are called **transgression functors**.
In order to apply transgression, we need to equip the Chern-Simons 2-gerbe $\mathcal{C}S_M$ with a connection. This can be done in a canonical way using the connection on the basic gerbe $\mathcal{G}_{bas}$ of curvature $H(X, Y, Z) = \langle X, [Y, Z] \rangle$, and the Chern-Simons 3-form

$$\langle A \wedge dA \rangle + \frac{2}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(FM).$$

where $A$ is the Levi-Civita connection 1-form on $FM$.

In order to transgress trivializations, we also need to equip them with connections; these are called string connections.

**Theorem (KW '09)**

Every string structure admits a string connection, and the set of string connections is affine.
A geometric string structure is a pair of a string structure and a string connection.

**Theorem (KW '09)**

The transgression of $CS_M$ to the loop space is the spin lifting gerbe $S_{LM}$. In particular, every geometric string structure on $M$ gives a spin structure on $LM$.

**Theorem (Bunke '10)**

The transgression of $CS_M$ to the mapping space $M^\Sigma$ is $\text{Pfaff}(\mathcal{D})$. In particular, every geometric string structure gives a trivialization of $\text{Pfaff}(\mathcal{D})$.

Conclusion: geometric string structures cancel the anomaly of the supersymmetric sigma model.
Remark 1 – Classification of string structures

- The set of isomorphism classes of string structures on a string manifold $M$ is parameterized by $\mathbb{H}^3(M, \mathbb{Z})$.
- The set of isomorphism classes of geometric string structures on a string manifold $M$ is parameterized by the differential cohomology group $\hat{\mathbb{H}}^3(M, \mathbb{Z})$.

Recall that $\hat{\mathbb{H}}^3(M, \mathbb{Z})$ is the group of B-fields on $M$, i.e. B-fields act on the geometric string structures. In particular, 2-forms $B \in \Omega^2(M)$ act on the string connections.

Under this action, the trivialization of $\text{Pfaff}(\mathcal{D})$ changes by

$$\exp 2\pi i \int_{\Sigma} B.$$  

In particular, it depends on the choice of the string connection.
Remark 2 – The covariant derivative of a string connection

Every geometric string structure on \( M \) determines a 3-form \( K \in \Omega^3(M) \) with \( dK = \frac{1}{2} \langle F_A \wedge F_A \rangle \).

The B-field action of \( B \in \Omega^2(M) \) takes \( K \) to \( K + dB \).

The Pfaffian \( Pfaff(\mathring{\mathcal{D}}) \) comes equipped with the Bismut-Freed connection. The section of \( Pfaff(\mathring{\mathcal{D}}) \) has covariant derivative

\[
\int_{\Sigma} ev^* K \in \Omega^1(M^\Sigma).
\]

Höhn-Stolz conjecture: if \( \text{Ric}_g > 0 \) and \( K = 0 \), then the Witten genus of \( M \) vanishes in \( tmf^{-n}(pt) \).
Remark 3 – The string 2-group

String structures can also be understood in terms of a (higher) reduction problem in non-abelian gerbes. There is a central extension

\[ \text{BU}(1) \to \text{String}(n) \to \text{Spin}(n) \]

of Lie 2-groups, and one can try to “reduce” the frame bundle \( FM \) to a non-abelian gerbe with structure 2-group \( \text{String}(n) \).

Theorem (KW-Nikolaus ’12)

The Chern-Simons 2-gerbe is the (higher) lifting gerbe of this reduction problem, i.e. there is a 1:1 correspondence between string structures and reductions of \( FM \) to \( \text{String}(n) \).
**Remark 4 – Spin structures on loop spaces revisited**

Recall: transgression takes string structures on $M$ to spin structures on $LM$.

The problem is that transgression is neither injective nor surjective. We have to describe the image of transgression.

**Theorem (KW ’14)**

*There is a 1:1 correspondence between string structures on $M$ and spin structures on $LM$ equipped with fusion product and thin homotopy equivariance.*
Summary:

- A string structure is higher geometrical structure whose existence is obstructed by $\frac{1}{2}p_1(M) \in \mathbb{H}^4(M, \mathbb{Z})$.

- Together with a string connection, it defines a trivialization of the Pfaffian line bundle of a family of Dirac operators parameterized by a space of maps $M^\Sigma$.

- The integrand of the path integral of the supersymmetric sigma model with target $M$ is a section in that Pfaffian bundle.

Given a geometric string structure it becomes a smooth map, i.e. the model becomes anomaly-free.
References


T. Nikolaus and K. Waldorf, “Lifting problems and transgression for non-abelian gerbes”.
[arxiv:1112.4702]

D. Stevenson, “Bundle 2-gerbes”.
[arxiv:math/0106018]

K. Waldorf, “String connections and Chern-Simons theory”.
[arxiv:0906.0117]

K. Waldorf, “String geometry vs. spin geometry on loop spaces”.
[arxiv:1403.5656]