Geometric string structures and supersymmetric sigma models

Konrad Waldorf

These are notes of a talk given at the workshop "Higher Gauge Theory, TQFT's and Categorification" at Cardiff University in May 2011

1 Motivation

Setup for a 2-dimensional, bosonic sigma model:

- target space: Riemannian manifold M
- worldsheet: Riemann surface Σ
- fields: smooth maps $\phi : \Sigma \longrightarrow M$
- action functional:

$$S^{bos}(\phi) := \int_{\Sigma} \langle \mathrm{d}\phi \wedge \star \mathrm{d}\phi \rangle$$

Setup for the supersymmetric sigma model:

- require additionally a spin structure on Σ.
- for each field $\phi : \Sigma \longrightarrow M$, there is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}_{\phi}$ of fermionic fields, and a Dirac operator

$$D\!\!\!/_{\phi} : \mathcal{H}_{\phi}^{+} \longrightarrow \mathcal{H}_{\phi}^{+}.$$

- additional term in the action functional

$$S_{\phi}^{fer}(\psi) := \int_{\Sigma} \left\langle \psi, D\!\!\!/_{\phi}\psi \right\rangle \, \mathrm{dvol}_{\Sigma}.$$

Problem: give sense to the "quantum integrand"

$$\mathcal{A}^{susy}(\phi) = \exp\left(S^{bos}(\phi)\right) \cdot \int_{\mathcal{H}_\phi^+} \exp\left(S_\phi^{fer}(\psi)\right) \mathrm{d}\psi$$

as a *smooth function*

$$\mathcal{A}^{susy} : C^\infty(\Sigma, M) \longrightarrow \mathbb{C}.$$

In this talk, I want to describe a solution to this problem based on work of Freed [Fre87], Freed-Moore [FM06], Bunke [Bun] and myself [Wal]. Overview:

- Using theory of differential operators one defines a line bundle $\mathcal{P}\mathit{faff}(\not{D}_M)$ over $C^\infty(\Sigma, M)$ together with a smooth section $\mathrm{pfaff} : C^\infty(\Sigma, M) \longrightarrow \mathcal{P}\mathit{faff}(\not{D}_M)$. Upon interpreting the fermionic path integral as a Berezinian integral, one gets

$$\int_{\mathcal{H}_\phi^+} \exp\left(S_\phi^{fer}(\psi)\right) \mathrm{d}\psi = \mathrm{pfaff}(\phi).$$

- A geometric string structure $\mathbb{T}$ on M defines a trivialization $t_\mathbb{T} : \mathcal{P}\mathit{faff}(\not{D}_M) \longrightarrow \mathbb{C}$. The composition of the trivialization $t_\mathbb{T}$ with the section pfaff defines the desired smooth function,

$$\mathcal{A}_\mathbb{T}^{susy} := \exp(S^{bos}) \cdot (t_\mathbb{T} \circ \mathrm{pfaff}).$$

2 Determinant and Pfaffian Bundles

Linear algebra:

- Let V_0 and V_1 be finite-dimensional vector spaces, and $f : V_0 \longrightarrow V_1^*$ be a linear map. Taking highest exterior powers yields the determinant $\det(f) : \det V_0 \longrightarrow \det V_1^*$, which can be regarded as an element

$$\det(f) \in \det V_0^* \otimes \det V_1^*.$$

- Suppose $V_0 = V_1 =: V$ and $\dim V = 2n$. The map $f : V \longrightarrow V^*$ is skew-symmetric if it corresponds to an element $f \in \Lambda^2 V^*$. In this case, its Pfaffian is defined by

$$\mathrm{pfaff}(f) := \frac{1}{n!} f^n \in \Lambda^{2n} V^* = \det V^*.$$

We have

$$\mathrm{pfaff}(f) \otimes \mathrm{pfaff}(f) = \det(f)$$

as elements of $\det V^* \otimes \det V^*$.

Consider an odd self-adjoint elliptic operator $D : \mathcal{H} \longrightarrow \mathcal{H}$ acting on a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}$, i.e. its spectrum is real, discrete and the eigenspaces are graded finite-dimensional. The spectrum of the operators $D_\pm^2$ is then positive, and still discrete with graded finite-dimensional eigenspaces.

- We define for $0 \leq \lambda < \mu$ the finite-dimensional vector spaces

  $$\mathcal{H}_\pm^{\lambda,\mu} := \bigoplus_{\lambda \leq \epsilon < \mu} \mathrm{Eig}(D_\pm^2, \epsilon);$$

  Notice that

  $$\mathcal{H}_\pm^{\lambda,\mu} \cong \mathcal{H}_\pm^{\lambda,\epsilon} \oplus \mathcal{H}_\pm^{\epsilon,\mu}.$$

  The operator D restricts to a linear operator

  $$D_\pm^{\lambda,\mu} := D|_{\mathcal{H}_\pm^{\lambda,\mu}} : \mathcal{H}_\pm^{\lambda,\mu} \longrightarrow \mathcal{H}_\mp^{\lambda,\mu}.$$

- We define

  $$\mathcal{H}^{\lambda,\mu} := \mathcal{H}_+^{\lambda,\mu} \oplus \left(\mathcal{H}_-^{\lambda,\mu}\right)^*.$$

  Then, we have

  $$\det D_+^{\lambda,\mu} \in \det(\mathcal{H}_+^{\lambda,\mu})^* \otimes \det(\mathcal{H}_-^{\lambda,\mu}) = \det \mathcal{H}^{\lambda,\mu}.$$

- Now we suppose that $J : \mathcal{H} \longrightarrow \mathcal{H}$ is an odd, anti-linear, anti-self-adjoint isomorphism that commutes with D. Anti-linear means it is linear as a map $\mathcal{H} \longrightarrow \overline{\mathcal{H}}$ to the opposed vector space. We define the linear anti-self-adjoint operator

  $$\not{D}^{\lambda,\mu} := J_- \circ D_+^{\lambda,\mu} : \mathcal{H}_+^{\lambda,\mu} \longrightarrow \overline{\mathcal{H}_+^{\lambda,\mu}}.$$

  Then consider the skew-symmetric operator

  $$\alpha^{\lambda,\mu} : \mathcal{H}_+^{\lambda,\mu} \longrightarrow (\mathcal{H}_+^{\lambda,\mu})^* \quad \text{with} \quad \alpha^{\lambda,\mu}(\psi)(\varphi) := \left\langle \psi, \not{D}^{\lambda,\mu}(\varphi) \right\rangle,$$

  which we regard as an element $\alpha^{\lambda,\mu} \in \Lambda^2(\mathcal{H}_+^{\lambda,\mu})^*$. Its pfaffian is denoted

  $$\mathrm{pfaff}^{\lambda,\mu} \in \det(\mathcal{H}_+^{\lambda,\mu})^*.$$

Now we consider a *family* of odd self-adjoint elliptic operators D_b parameterized by a (possibly infinite-dimensional manifold) B.

- We define for $\mu \geq 0$ the open sets
$$U_\mu := \left\{ b \in B \mid \mu \notin \mathrm{spec}(D_b^2) \right\}.$$
For each $0 \leq \lambda < \mu$ the vector spaces $\mathcal{H}_{b,\pm}^{\lambda,\mu}$ form smooth vector bundles over U_μ. The elements $\det D_{b,+}^{\lambda,\mu}$ form a smooth section of $\det \mathcal{H}^{\lambda,\mu}$. The elements $\mathrm{pfaff}_b^{\lambda,\mu}$ form a smooth section of $\det(\mathcal{H}_+^{\lambda,\mu})^*$.

- We have over U_μ the line bundle $\det \mathcal{H}^{0,\mu}$ and over $U_\mu \cap U_\nu$ with $\nu > \mu$ the isomorphism
$$\mathrm{id} \otimes \det D_+^{\mu,\nu} : \det \mathcal{H}^{0,\mu} \longrightarrow \det \mathcal{H}^{0,\mu} \otimes \det \mathcal{H}^{\mu,\nu} = \det \mathcal{H}^{0,\nu}.$$
The determinant line bundle $\mathcal{D}et(D)$ over B is glued from this data. The local sections $\det D_+^{0,\mu}$ glue to a global smooth section det of $\mathcal{D}et(D)$.

- We have over U_μ the line bundle $\det(\mathcal{H}_+^{0,\mu})^*$ and over $U_\mu \cap U_\nu$ with $\nu > \mu$ the isomorphism
$$\mathrm{id} \otimes \mathrm{pfaff}^{\mu,\nu} : \det(\mathcal{H}_+^{0,\mu})^* \longrightarrow \det(\mathcal{H}_+^{0,\mu})^* \otimes \det(\mathcal{H}_+^{\mu,\nu})^* = \det(\mathcal{H}_+^{0,\nu})^*.$$
The Pfaffian line bundle $\mathcal{P}\mathit{faff}(\not{D})$ over B is glued from this data. The local sections $\mathrm{pfaff}^{0,\mu}$ glue to a global smooth section pfaff of $\mathcal{P}\mathit{faff}(\not{D})$.

- There is an isomorphism
$$\mathcal{P}\mathit{faff}(\not{D}) \otimes \mathcal{P}\mathit{faff}(\not{D}) \cong \mathcal{D}et(D)^*$$
of line bundles over B which is over U_μ given by
$$\mathrm{id} \otimes \det J|_{\det(\mathcal{H}_+^{0,\mu})^*} : \det(\mathcal{H}_+^{0,\mu})^* \otimes \det(\mathcal{H}_+^{0,\mu})^* \longrightarrow \det(\mathcal{H}^{0,\mu})^*.$$
Under this isomorphism, the section $\mathrm{pfaff} \otimes \mathrm{pfaff}$ corresponds to the section det.

Geometric data on the bundles $\mathcal{D}et(D)$ and $\mathcal{P}\mathit{faff}(\not{D})$:

- The determinant bundle $\mathcal{D}et(D)$ comes equipped with a hermitian metric, the "Quillen metric", and a unitary connection, the "Bismut-Freed connection".

- Via the isomorphism $\mathcal{P}\mathit{faff}(\not{D}) \otimes \mathcal{P}\mathit{faff}(\not{D}) \cong \mathcal{D}et(D)$ one induces metric and connection on $\mathcal{P}\mathit{faff}(\not{D})$.

3 The Quantum Integrand

Linear algebra:

- Let V be a finite-dimensional vector space, $\dim V = 2n$. The *Berezinian* is the linear map
$$\int^B : \Lambda^* V^* \longrightarrow \det V^*$$
which is defined on monomials $\alpha \in \Lambda^k V^*$ by
$$\int^B \alpha = \begin{cases} \alpha & \text{if } k = 2n \\ 0 & \text{else.} \end{cases}$$
Remark: usually, if V has an orientation $\omega \in \det V$, one understands the Berezinian as the composition of the one above with the pairing $\det V^* \longrightarrow \mathbb{K} : \alpha \longmapsto \alpha(\omega)$.

- For $\alpha \in \Lambda^2 V^*$ we have:
$$\int^B \exp(\alpha) = \mathrm{pfaff}(\alpha).$$

Return to the situation of the supersymmetric sigma model.

- The parameter space is $B := C^\infty(\Sigma, M)$.

- The spinor bundle $S(\Sigma)$ is $\mathbb{Z}/2\mathbb{Z}$-graded and has by dimensional reasons an odd quaternionic structure.

- We let W be the real $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over M with $W_+ := TM$ and W_- the trivial bundle of rank $\dim M$. For each $\phi \in B$, we have a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V := S(\Sigma) \otimes_{\mathbb{R}} \phi^* W$ over Σ. Notice that $V_+ \cong V_- \cong S(\Sigma) \otimes_{\mathbb{R}} \phi^* TM$.

- Since W is a real vector bundle, the quaternionic structure of $S(\Sigma)$ extends to V. Furthermore, since $S(\Sigma)$ and W carry connections induced from the Levi-Civita connections on Σ and M, respectively, V carries a connection.

- For $\phi \in B$ we have the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}_\phi := \Gamma(V)$ of smooth sections into V, equipped with the L_2 scalar product
$$(\psi, \varphi) \longmapsto \int_\Sigma \langle \psi, \varphi \rangle \, \mathrm{dvol}_\Sigma.$$

- The Dirac operator

$$D_\phi : \mathcal{H}_\phi \longrightarrow \mathcal{H}_\phi$$

is given as usual by the covariant derivative in V followed by Clifford multiplication on $S(\Sigma)$. The quaternionic structure on V defines a quaternionic structure J_ϕ on $\mathcal{H}_\phi$.

- Now consider $\mu \geq 0$ and $\phi \in U_\mu$. We have the finite-dimensional vector space $\mathcal{H}^{0,\mu}_{\phi,+}$ and the 2-form $\alpha^{0,\mu}_\phi \in \Lambda^2(\mathcal{H}^{0,\mu}_{\phi,+})^*$. Inserting the definition of $\alpha^{0,\mu}_\phi$ we have the well-defined equality

$$\int^B \exp\left(\int_\Sigma \left\langle -, \not{D}^{0,\mu}_\phi - \right\rangle \, \mathrm{dvol}_\Sigma\right) = \mathrm{pfaff}(\phi).$$

Since the right hand side is independent of μ, the left hand side is also independent. Dropping the index μ, and adding some ψ's as extra notation produces exactly the fermionic path integral.

4 Geometric String Structures

A spin manifold M is called *string manifold*, if the class

$$\frac{1}{2}p_1(TM) \in \mathrm{H}^4(M, \mathbb{Z})$$

vanishes. Overview:

1. Realize the class $\frac{1}{2}p_1(TM)$ as a geometrical object over M, the "Chern-Simons 2-gerbe" $\mathbb{CS}_M$.

2. Define a *string structures* on M as a trivialization of $\mathbb{CS}_M$. Thus, $\frac{1}{2}p_1(TM) = 0$ if and only if M admits a string structure.

3. The Chern-Simons 2-gerbe $\mathbb{CS}_M$ carries a canonical connection defined from the Riemannian metric on M. Define a *geometric string structure* on M as connection-preserving trivialization of $\mathbb{CS}_M$.

Construction of the Chern-Simons 2-gerbe $\mathbb{CS}_M$:

- To construct a bundle 2-gerbe, we need a surjective submersion. For $\mathbb{CS}_M$ we take the spin frame bundle $P_{\mathrm{Spin}(n)} \longrightarrow M$.

- Next we need a bundle gerbe over the 2-fold fibre product $P_{\mathrm{Spin}(n)}^{[2]}$. Notice that there is a smooth map
$$g : P_{\mathrm{Spin}(n)}^{[2]} \longrightarrow \mathrm{Spin}(n)$$
the "universal transition function". We take the pullback of the basic gerbe $\mathcal{G}$ over $\mathrm{Spin}(n)$ along g.

- Finally we need coherence structure over the higher fibre products of $P_{\mathrm{Spin}(n)}$. This structure is provided by a multiplicative structure on $\mathcal{G}$.

- The calculation that $c_3(\mathbb{CS}_M) = \frac{1}{2}p_1(TM)$ uses that the characteristic class of the multiplicative basic gerbe $\mathcal{G}$ in $\mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z})$ is the universal class $\frac{1}{2}p_1$.

- For the connection on $\mathbb{CS}_M$, we need a 3-form on $P_{\mathrm{Spin}(n)}$. We take the Chern-Simons 3-form associated to the Levi-Cevita connection A on $P_{\mathrm{Spin}(n)}$:
$$CS(A) := \langle A \wedge \mathrm{d}A \rangle + \frac{2}{3} \langle A \wedge A \wedge A \rangle \in \Omega^3(P_{\mathrm{Spin}(n)}).$$
The remaining structure for the connection is provided by a connection on the basic gerbe $\mathcal{G}$. The curvature of the connection on $\mathbb{CS}_M$ is the Pontryagin 4-form
$$\mathrm{curv}(\mathbb{CS}_M) = \frac{1}{2} \langle F_A \wedge F_A \rangle \in \Omega^4(M).$$

Trivializations of $\mathbb{CS}_M$, i.e. string structures on M:

- A trivialization $\mathbb{T}$ of $\mathbb{CS}_M$ is a bundle gerbe $\mathcal{S}$ over $P_{\mathrm{Spin}(n)}$ together with an isomorphism
$$\mathcal{A} : g^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{S} \longrightarrow \mathrm{pr}_1^*\mathcal{S}$$
of bundle gerbes over $P_{\mathrm{Spin}(n)}^{[2]}$ plus coherence structure on the higher fibre products. Remark: The isomorphism $\mathcal{A}$ restricts over each fibre $F \cong \mathrm{Spin}(n)$ of $P_{\mathrm{Spin}(n)}$ to an isomorphism $\mathcal{S}|_F \cong \mathcal{G}$. In particular,
$$c_3(\mathcal{S})|_F = 1 \in \mathbb{Z} = \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z}).$$
This reproduces the definition of s string structure given by Stolz and Teichner [ST04].

- Being connection-preserving is actually additional structure for trivializations of 2-gerbes, not just a property. Namely, it is a connection on the gerbe $\mathcal{S}$ such that $\mathcal{A}$ is connection-preserving.

- A connection-preserving trivialization $\mathbb{T}$ determines a 3-form $H_{\mathbb{T}} \in \Omega^3(M)$ by

$$\mathrm{pr}^* H_{\mathbb{T}} = \mathrm{curv}(\mathcal{S}) + CS(A).$$

It satisfies $\mathrm{d}H_{\mathbb{T}} = \frac{1}{2} \langle F_A \wedge F_A \rangle$.

Action by gerbes:

- If $\mathbb{T} = (\mathcal{S}, \mathcal{A})$ is a string structure and $\mathcal{K}$ is a bundle gerbe over M, there is a new string structure $\mathbb{T} \otimes \mathcal{K}$ defined by $\mathcal{S}' := \mathcal{S} \otimes \mathrm{pr}^*\mathcal{K}$ and $\mathcal{A}' := \mathcal{A} \otimes \mathrm{id}_{\mathrm{pr}^*\mathcal{K}}$. This action is simply transitive on equivalence classes.
- If $\mathbb{T}$ is a *geometric* string structure, and $\mathcal{K}$ is a bundle gerbe *with connection* over M, then $\mathbb{T} \otimes \mathcal{K}$ is again a geometric string structure. The 3-forms satisfy

$$H_{\mathbb{T}\otimes\mathcal{K}} = H_{\mathbb{T}} \otimes \mathrm{curv}(\mathcal{K}).$$

5 Transgression

Suppose $\mathbb{G}$ is a 2-gerbe with connection over M, and Σ is a closed oriented surface.

- For a smooth map $\phi : \Sigma \longrightarrow M$, define the set T_ϕ of (equivalence classes of) connection-preserving trivializations of $\phi^*\mathbb{G}$. Via the action

$$\mathbb{T} \longmapsto \mathbb{T} \otimes \mathcal{K}$$

this is a torsor over the group of isomorphism classes of gerbes with (necessarily flat) connection over Σ, which are classified by $\mathrm{H}^2(\Sigma, \mathrm{U}(1)) \cong \mathrm{U}(1)$. The $\mathrm{U}(1)$-torsors T_ϕ fit together to a Fréchet principal $\mathrm{U}(1)$-bundle $\mathscr{T}_{\mathbb{G}}$ over $C^\infty(\Sigma, M)$.
- A connection on $\mathscr{T}_{\mathbb{G}}$ is obtained from the parallel transport in the 2-gerbe $\mathbb{G}$.
- In differential cohomology, the assignment $\mathbb{G} \longmapsto \mathscr{T}_{\mathbb{G}}$ realizes the transgression homomorphism

$$\hat{\mathrm{H}}^4(M, \mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^2(C^\infty(\Sigma, M)).$$

Applied to the Chern-Simons 2-gerbe we get:

- A principal U(1)-bundle $\mathscr{T}_{\mathbb{CS}_M}$ over $C^\infty(\Sigma, M)$ with connection.
- A geometric string structure (i.e. a connection-preserving trivialization $\mathbb{T}$ of $\mathbb{CS}_M$) defines a global smooth section

$$s_{\mathbb{T}} : C^\infty(\Sigma, M) \longrightarrow \mathscr{T}_{\mathbb{CS}_M} : \phi \longmapsto \phi^*\mathbb{T}.$$

- Fact 1: if $\omega \in \Omega^1(\mathscr{T}_{\mathbb{CS}_M})$ is the connection 1-form on $\mathscr{T}_{\mathbb{CS}_M}$, then

$$s_{\mathbb{T}}^*\omega = \int_\Sigma \mathrm{ev}^* H_{\mathbb{T}}.$$

- Fact 2: if $\mathcal{K}$ is a bundle gerbe with connection over M, then

$$s_{\mathbb{T}\otimes\mathcal{K}} = s_{\mathbb{T}} \cdot \mathrm{Hol}_{\mathcal{K}}.$$

Bunke constructs [Bun] a connection-preserving isomorphism

$$B : \mathscr{T}_{\mathbb{CS}_M} \longrightarrow \mathcal{P}\mathit{faff}(\not{D}_M).$$

Outline of the construction:

- We work over a fixed point $\phi : \Sigma \longrightarrow M$. Let $\varphi : \phi^* P_{\mathrm{Spin}(n)} \longrightarrow \mathrm{Spin}(n)$ be a trivialization of the spin frame bundle of M. By functorality, it induces a trivialization $\mathbb{T}_\varphi$ of $\phi^*\mathbb{CS}_M$, namely the one with $\mathcal{S}_\varphi := \varphi^*\mathcal{G}$ and $\mathcal{A}$ given by the multiplicative structure on $\mathcal{G}$.
- We look at the family of (generalized) Dirac operators parameterized by $\mathbb{R}$, which is over $t \in \mathbb{R}$

$$D_t^\varphi = D_\phi + 1 \otimes tQ^\varphi \quad \text{with} \quad Q^\varphi = \begin{pmatrix} 0 & \varphi^* \\ \varphi & 0 \end{pmatrix},$$

  where φ is considered as a trivialization $\phi^*TM \longrightarrow \mathbf{I}$ of the tangent bundle. The associated Pfaffian bundle over $\mathbb{R}$ is denoted $\mathcal{P}\mathit{faff}(\not{D}^\varphi)$; it comes with its section $\mathrm{pfaff}(\not{D}^\varphi)$.
- The Laplacian of D_t^φ is

$$(D_t^\varphi)^2 = D_\phi^2 + tD_\phi Q^\varphi + t^2(Q^\varphi)^2.$$

  The t^2-term is dominating, and so there exists $t_0 \geq 0$ such that D_t^φ over $[t_0, \infty)$ is positive, in particular invertible. Thus, $\mathrm{pfaff}(\not{D}^\varphi)(t) \neq 0$ for all $t \geq t_0$.

- For $x \geq t_0$ consider the element

$$s_x^\varphi(t) := pt^\nabla_{\gamma_{x,t}}(\mathrm{pfaff}(\not{D}^\varphi(x))),$$

where pt^∇ denotes the parallel transport in $\mathcal{P}\mathit{faff}(\not{D}^\varphi)$, and $\gamma_{x,t}$ is the canonical path in $\mathbb{R}$ from x to t. Since parallel transport is an isometry, $s_x^\varphi(t)$ is non-zero and can be normalized to unit length.

- The limit

$$s^\varphi(t) := \lim_{x\to\infty} s_x^\varphi(t)$$

exists in a certain sense and defines a smooth nowhere vanishing section of $\mathcal{P}\mathit{faff}(\not{D}^\varphi)$. In particular,

$$0 \neq s^\varphi(0) \in \mathcal{P}\mathit{faff}(\not{D}_M).$$

- Define the isomorphism by

$$B : \mathscr{T}_{\mathbb{CS}_M} \longrightarrow \mathcal{P}\mathit{faff}(\not{D}_M) : \mathbb{T} \longmapsto s^\varphi(0) \cdot \mathrm{Hol}_{\mathcal{K}}(\Sigma),$$

where φ is some choice of a trivialization, and $\mathcal{K}$ is a bundle gerbe with connection over Σ such that $\phi^*\mathbb{T} = \mathbb{T}_\varphi \otimes \mathcal{K}$.

6 Concluding Remarks

1. The section $s_{\mathbb{T}}$ of $\mathscr{T}_{\mathbb{CS}_M}$ depends on the *geometric* part of the string structure $\mathbb{T}$. In order to see this, consider a 2-form $\eta \in \Omega^2(M)$ and the trivial bundle gerbe $\mathcal{I}_\eta$ with connection η. Then, $\mathbb{T} \otimes \mathcal{I}_\eta$ has the same underlying string structure as $\mathbb{T}$. However, we have

$$s_{\mathbb{T}\otimes\mathcal{I}_\eta}(\phi) = s_{\mathbb{T}}(\phi) \cdot \exp\left(\int_\Sigma \phi^*\eta\right).$$

2. Wess-Zumino term: this is an additional term in the exponentiated action of the *bosonic* sigma model. A Wess-Zumino term is given by fixing a bundle gerbe $\mathcal{K}$ with connection over M, then

$$\mathcal{A}_{\mathcal{K}}^{WZ}(\phi) := \mathrm{Hol}_{\mathcal{K}}(\phi).$$

In the supersymmetric context, it can be compensated in the fermionic part of the action:

$$\mathcal{A}_{\mathbb{T}}^{susy} \cdot \mathcal{A}_{\mathcal{K}}^{WZ} = \mathcal{A}_{\mathbb{T}\otimes\mathcal{K}}^{susy}.$$

3. Heuristical considerations with path integrals suggest that the fermionic path integral requires a gauge fixing [FM06]. Gauge fixing introduces new terms to the action of the supersymmetric sigma model.

4. In M-theory there are yet more terms. In particular, the Dirac operator on Σ is not twisted by the tangent bundle TM alone, but rather by a tensor product of TM and some principal E_8-bundle P over M.

5. For $\Sigma = S^1 \times S^1$ the torus, spin structures on Σ are determined by complex structures, which in turn are determined by a complex number $q \in \mathbb{C}$. The assignment

$$q \longmapsto \int_{C^\infty(\Sigma_q, M)} \mathcal{A}^{susy}(\phi)\mathrm{d}\phi$$

is supposed to be a modular form that represents the Witten genus of the target manifold M.

References

[Bun] U. Bunke, *String Structures and Trivialisations of a Pfaffian Line Bundle*, preprint. `[arxiv:0909.0846]`

[FM06] D. S. Freed and G. W. Moore, *Setting the Quantum Integrand of M-Theory*, Commun. Math. Phys. **263** (1), 89–132 (2006). `[arxiv:hep-th/0409135]`

[Fre87] D. S. Freed, On Determinant Line Bundles, in *Mathematical Aspects of String Theory*, edited by S. T. Yau, World Scientific, 1987.

[ST04] S. Stolz and P. Teichner, *What is an elliptic Object?*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343, Cambridge Univ. Press, 2004.

[Wal] K. Waldorf, *String Connections and Chern-Simons Theory*, preprint. `[arxiv:0906.0117]`