Differential string classes and loop spaces

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Abstract

String classes are secondary characteristic classes for spin manifolds. They stand for the cancellation of a certain anomaly in fermionic string theories. I will describe a refinement of string classes to differential cohomology, which more appropriately implements this cancellation mechanism. The definition of differential string classes will be motivated using an approach with connections on gerbes and 2-gerbes. I will then explain an alternative but equivalent perspective in terms of ordinary differential geometry on the free loop space.

Basic setup and notation:

- M a spin manifold of dimension n
- $\pi : FM \longrightarrow M$ the spin-oriented frame bundle of M
- for $p \in FM$ get inclusion $i_p : \mathrm{Spin}(n) \longrightarrow FM : g \longmapsto pg$
- $\gamma \in \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$ a fixed generator

Definition: A **string class** on M is a class $\xi \in \mathrm{H}^3(FM, \mathbb{Z})$ such that $i_p^*\xi = \gamma$ for all $p \in FM$.

The Serre spectral sequence gives an exact sequence

$$0 \longrightarrow \mathrm{H}^3(M,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(FM,\mathbb{Z}) \xrightarrow{i_p^*} \mathrm{H}^3(\mathrm{Spin}(n),\mathbb{Z}) \xrightarrow{tr} \mathrm{H}^4(M,\mathbb{Z})$$

$$\gamma \longmapsto \tfrac{1}{2}p_1(M)$$

We conclude: string classes on M exist if and only if $\frac{1}{2}p_1(M) = 0$.
In this case, string classes form a torsor over $\mathrm{H}^3(M,\mathbb{Z})$.

Slogan: "String classes are trivializations of $\frac{1}{2}p_1(M)$"

What are string classes good for?

- Not too much, actually. What we really need are *differential* string classes.

So, how are differential string classes defined, and what are *they* good for?

- The definition requires some care; we'll motivate it below using a cocycle model, "geometric string structures".
- A differential string class on M cancels the "global worldsheet anomaly" of supersymmetric sigma models on M.

1.) Geometric string structures

2.) Differential string classes

3.) Spin geometry of loop spaces

The Stolz-Teichner program represents cohomology classes by various types of field theories. Here:

$$TFT^3(M) := \left\{ \begin{array}{c} \text{3-dimensional, fully} \\ \text{extended topological} \\ \text{field theories over } M \end{array} \right\} \longrightarrow \mathrm{H}^4(M,\mathbb{Z})$$

$$\begin{array}{c} \text{Classical Chern-Simons} \\ \text{theory on } M \text{ at level } \gamma \end{array} \longmapsto \tfrac{1}{2}p_1(M)$$

Thus, Chern-Simons theory is a cocycle representative for $\frac{1}{2}p_1(M)$; accordingly, a geometric string structure should be a **trivialization of Chern-Simons theory**. This is the original definition of Stolz and Teichner.

We'll use another cocycle model for $\frac{1}{2}p_1(M)$, namely **bundle 2-gerbes with connection**. These define (invertible) field theories whose value on the point is the 2-gerbe, in such a way that we have a commutative diagram:

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{Bundle 2-gerbes with}\\ \text{connection over } M\end{array}\right\} = 2\text{-}\mathcal{G}rb^{\nabla}(M) & & \\
\downarrow & \searrow & \\
 & & \mathrm{H}^4(M,\mathbb{Z}) \\
 & \nearrow & \\
\left\{\begin{array}{c}\text{3-dimensional, fully}\\ \text{extended topological}\\ \text{field theories over } M\end{array}\right\} = TFT^3(M) & &
\end{array}$$

The relevant 2-gerbe is the Chern-Simons bundle 2-gerbe, and we'll construct it in the following.

Bundle *n*-gerbes are a recursive approach to "higher" bundles:

- Bundle: an open cover, transition functions $g_{\alpha\beta}$ and local connection 1-forms A_α such that

$$(\delta A)_{\alpha\beta} := A_\beta - A_\alpha = g_{\alpha\beta}^{-1} \mathrm{d} g_{\alpha\beta}.$$

- Bundle gerbe: transition *bundles* with connections, and local connection *2*-forms.
- Bundle 2-gerbe: transition *bundle gerbes* with connections, and local connection *3*-forms.

  etc...

Bundle *n*-gerbes with connection over X are a cocycle model for (differential) cohomology:

$$n\text{-}\mathcal{G}rb^{\nabla}(M)_{/\,\mathsf{Iso}} \xrightarrow{\cong} \hat{\mathrm{H}}^{n+2}(M) \xrightarrow{c} \mathrm{H}^{n+2}(M, \mathbb{Z})$$

The **Chern-Simons bundle 2-gerbe** $\mathbb{CS}_M$ is defined as follows [Carey et al. '05, KW '08]:

- The *"open cover"* is the total space FM of the spin frame bundle, intersections correspond to fibre products
- The *"local connection 3-form"* is the Chern-Simons form of the Levi-Cevita connection A:

$$CS(A) = \langle A \wedge \mathrm{d}A \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(FM)$$

- The *"transition gerbe"* $\mathcal{H}$ over $FM \times_M FM$ is obtained by pulling back the basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(n)$ along the difference map $g : FM \times_M FM \longrightarrow \mathrm{Spin}(n)$, i.e.

$$\mathcal{H} := g^* \mathcal{G}_{bas}.$$

The difficult point is to find the correct connection on $\mathcal{H}$.

- The basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(n)$ carries a connection whose curvature is the Cartan-3-form of the Lie group $\mathrm{Spin}(n)$,

$$H \in \Omega^3(\mathrm{Spin}(n)).$$

We equip the transition bundle gerbe $\mathcal{H} = g^*\mathcal{G}_{bas}$ with the pullback connection, shifted by the 2-form

$$\omega := \left\langle g^{-1}\mathrm{d}g \wedge \mathrm{pr}_1^*A \right\rangle \in \Omega^2(FM \times_M FM).$$

- The compatibility condition between the connection on the transition gerbe $\mathcal{H}$ and the local connection 3-form $CS(A)$ is

$$\delta CS(A) = \mathrm{curv}(\mathcal{H}).$$

As $\mathrm{curv}(\mathcal{H}) = g^*H + \mathrm{d}\omega$ (by construction), this is a well-known property of the Chern-Simons 3-form.

The Chern-Simons 2-gerbe $\mathbb{CS}_M$ has the following properties:

- The corresponding fully extended 3-dimensional topological field theory is the Chern-Simons theory at level γ.

  Proof: The field theory is defined so that its value on a closed oriented 3-manifold $\phi : S \longrightarrow M$ is the 3-holonomy of the 2-gerbe $\mathbb{CS}_M$. That 3-holonomy is obtained by lifting ϕ to a map $\tilde{\phi} : S \longrightarrow FM$, pulling back the local connection 3-form, and integrating. This gives the usual Chern-Simons action.

- The underlying cohomology class is $\frac{1}{2} p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$.

  Proof (for real cohomology classes): the cohomology is represented by the curvature 4-form, which is the derivative of the local connection 3-form: $\mathrm{d}CS(A) = \frac{1}{2} \langle F_A \wedge F_A \rangle$.

Definition: A **geometric string structure** is a trivialization of $\mathbb{CS}_M$ with connection ("string connection").

Thus, a geometric string structure is a bundle gerbe $\mathcal{S}$ with connection over FM, together with a connection-preserving isomorphism $\delta\mathcal{S} \cong \mathcal{H}$.

- Geometric string structures form a bicategory that has the structure of a 2-torsor over the monoidal bicategory $\mathcal{G}rb^{\nabla}(M)$.
- Underlying string class $\xi := c([\mathcal{S}]) \in \mathrm{H}^3(FM, \mathbb{Z})$.
  Proof: the isomorphism $\delta\mathcal{S} \cong \mathcal{H}$ implies $\delta\xi = g^*\gamma$. Pullback along $j_p : \mathrm{Spin}(n) \longrightarrow FM \times_M FM : g \longmapsto (gp, p)$ gives the condition $i_p^*\xi = \gamma$.
- **Theorem** [KW '09]: Every string class can be represented by a geometric string structure.

1.) Geometric string structures

2.) Differential string classes

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Definition: A **differential string class** is a class $\hat{\xi} \in \hat{\mathrm{H}}^3(FM)$ such that

$$\delta\xi = \hat{\omega} + g^*\hat{\gamma},$$

where $\hat{\omega}$ is the differential cohomology class of the 2-form ω, and $\hat{\gamma}$ is the differential cohomology class of $\mathcal{G}_{bas}$ $(c(\hat{\gamma}) = \gamma)$.

Theorem [KW '14]: The map $\mathcal{S} \longmapsto [\mathcal{S}] \in \hat{\mathrm{H}}^3(FM)$ is a bijection between isomorphism classes of geometric string structures and differential string classes. In particular:

- Differential string classes exist if and only if $\frac{1}{2}p_1(M) = 0$.
- Differential string classes form a torsor over $\hat{\mathrm{H}}^3(M)$.
- The map from differential string classes to string classes is surjective and each fibre is a torsor over $\Omega^2(M)/\Omega^2_{cl,\mathbb{Z}}(M)$.
  Proof: It is a map from a $\hat{\mathrm{H}}^3(M)$-torsor to a $\mathrm{H}^3(M,\mathbb{Z})$-torsor.

Remark: The sequence we got from the Serre spectral sequence exists in differential cohomology:

$$0 \longrightarrow \hat{\mathrm{H}}^3(M) \xrightarrow{\pi^*} \hat{\mathrm{H}}^3(FM) \xrightarrow{res} \hat{\mathrm{H}}^3(\mathrm{Spin}(n)) \xrightarrow{tr} \hat{\mathrm{H}}^4(M)$$
$$\hat{\gamma} \longmapsto \widehat{\tfrac{1}{2}p_1(M)}$$

However, the sequence is not exact, and repeating the definition of a string class "with hats everywhere" would give a wrong result. More specifically, for a class $\hat{\xi} \in \hat{\mathrm{H}}^3(FM)$

$$\delta\hat{\xi} = \hat{\omega} + g^*\hat{\gamma} \quad \Longrightarrow \quad i_p^*\hat{\xi} = \hat{\gamma}$$

(as $i_p^*\omega = 0$) but the converse in in general not true.

Differential cohomology has a **fibre integration map** [Gomi-Terashima '01, Bär-Becker '13].

If Σ is a closed oriented surface, and $ev : C^\infty(\Sigma, M) \times \Sigma \longrightarrow M$ is the evaluation map, we obtain a homomorphism

$$\int_\Sigma ev^* : \hat{\mathrm{H}}^4(M) \longrightarrow \hat{\mathrm{H}}^2(C^\infty(\Sigma, M)).$$

This homomorphism lifts to our cocycle model, in terms of a *transgression functor*:

$$T_\Sigma : 2\text{-}\mathcal{G}rb^\nabla(M) \longrightarrow \mathcal{B}un^\nabla_{S^1}(C^\infty(\Sigma, M))$$

If Σ has a spin structure, one can consider a family D_ϕ of Dirac operators on Σ parameterized by $\phi \in C^\infty(\Sigma, M)$. It defines a Pfaffian line bundle *Pfaff*(D) over $C^\infty(\Sigma, M)$, which is equipped with the *Quillen metric* and the *Bismut-Freed connection*.

Theorem [Bunke '09]: There exists a canonical, metric and connection-preserving line bundle isomorphism

$$T_\Sigma(\mathbb{CS}_M) \times_{S^1} \mathbb{C} \cong \mathit{Pfaff}(D).$$

In particular, a geometric string structure defines a trivialization of *Pfaff*(D), which only depends on the isomorphism class of the geometric string structure. Thus, the last two theorems imply:

Corollary: A differential string class defines a trivialization of *Pfaff*(D).

The **supersymmetric sigma model** on Σ has the following fields:

- Maps $\phi \in C^\infty(\Sigma, M)$
- Spinors $\psi \in \Gamma(Sp(\Sigma) \otimes \phi^* TM)$

The fermionic path integral

$$\int_{\Gamma(Sp(\Sigma)\otimes\phi^* TM)} \exp\left(\int_\Sigma \langle\psi, D_\phi\psi\rangle \,\mathrm{dvol}_\Sigma\right) \mathrm{d}\psi$$

has a well-defined meaning as an element $p_\phi \in \mathit{Pfaff}(D_\phi)$, rather than a complex number. This is called the **global worldsheet anomaly**.

The elements p_ϕ form a smooth section $p \in \Gamma(\mathit{Pfaff}(D))$. Since a differential string class defines a trivialization of $\mathit{Pfaff}(D)$, it makes p a well-defined smooth *function* on $C^\infty(\Sigma, M)$. Thus, differential string classes *cancel* the global world sheet anomaly.

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Let $LX := C^\infty(S^1, X)$. We have the homomorphism

$$\int_{S^1} ev^* : \hat{\mathrm{H}}^k(X) \longrightarrow \hat{\mathrm{H}}^{k-1}(LX).$$

It lifts to our cocycle model of bundles and bundle gerbes; e.g. at $k = 3$ we have a *transgression functor* [Brylinski '92]

$$T_{S^1} : \mathcal{G}rb^\nabla(X) \longrightarrow \mathcal{B}un^\nabla_{S^1}(LX).$$

Example: The transgression of the basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(n)$ is a S^1-bundle $T_{S^1}(\mathcal{G}_{bas})$ over $L\mathrm{Spin}(n)$. It can be equipped with a multiplication in such a way that it constitutes a model for the universal central extension

$$1 \longrightarrow S^1 \longrightarrow \widetilde{L\mathrm{Spin}(n)} \longrightarrow L\mathrm{Spin}(n) \longrightarrow 1.$$

We transgress a geometric string structure $\mathcal{S}$:

- The transgression of the bundle gerbe $\mathcal{S}$ over FM is a S^1-bundle $\widetilde{FM} := T_{S^1}(\mathcal{S})$ over $LFM = F(LM)$.
- The isomorphism $\delta\mathcal{S} \cong g^*\mathcal{G}_{bas}$ induces a bundle morphism

$$\delta\widetilde{FM} \cong Lg^*\widetilde{L\mathrm{Spin}(n)}$$

  which says that the map $\widetilde{FM} \longrightarrow F(LM) \longrightarrow LM$ has the structure of a principal $\widetilde{L\mathrm{Spin}(n)}$-bundle over LM.
- In other words, $\widetilde{FM}$ lifts the structure group of $F(LM)$ from $L\mathrm{Spin}(n)$ to the universal central extension. Such lifts are called **spin structures on LM** [Killingback '87].
- The connection on $\widetilde{FM}$ becomes a *spin connection* in the sense of Coquereaux-Pilch.

Summarizing: differential string classes transgress to spin structures on LM and spin connections.

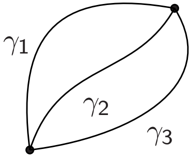
Killingback's idea was to use such data in order to trivialize the Pfaffian bundle $Pfaff(D)$. Indeed, one can show that the iterated transgression from LM to $LLM = C^\infty(T^2, M)$ of a spin structure on LM yields a trivialization of $Pfaff(D)$ over $C^\infty(T^2, M)$.

We'll see next that one can beef up spin structures on loop spaces so that they yields trivializations of $Pfaff(D)$ for *all* surfaces.

There is a category $\mathcal{F}us\mathcal{B}un_{S^1}^{\nabla_{sf}}(LX)$ consisting of S^1-bundles over LX with superficial connections and fusion products.

- A connection is **superficial** if its holonomy around a loop $\tau \in LLX$ vanishes whenever the rank of the associated torus $S^1 \times S^1 \longrightarrow X$ is not full.
- A **fusion product** on a S^1-bundle P over LX is an associative rule

$$P_{\gamma_1 \cup \gamma_2} \otimes P_{\gamma_2 \cup \gamma_3} \longrightarrow P_{\gamma_1 \cup \gamma_3}$$



where $\gamma_i \cup \gamma_j \in LX$ is obtained by concatenation of γ_i with the inverse of γ_j.

Theorem [KW '10]: Brylinski's transgression functor *lifts* to category of principal S^1-bundles with fusion products and superficial connections. This lift is an *equivalence of categories*.

$$\begin{array}{ccc} & & \mathcal{F}us\mathcal{B}un_{S^1}^{\nabla_{sf}}(LX) \\ & \overset{\cong}{\nearrow} & \downarrow \\ \mathcal{G}rb^{\nabla}(X) & \xrightarrow[T_{S^1}]{} & \mathcal{B}un_{S^1}^{\nabla}(LX) \end{array}$$

In particular, the central extension $\widetilde{L\mathrm{Spin}(n)}$ is equipped with a (multiplicative) fusion product.

A **fusion spin structure** is one whose S^1-bundle $\widetilde{FM}$ over LFM is equipped with a fusion product, and whose isomorphism $\delta\widetilde{FM} \cong Lg^* \widetilde{L\mathrm{Spin}(n)}$ is fusion-preserving. Obviously, all spin structures in the image of transgression are fusion.

Theorem [KW '14]: Transgression induces a bijection:

$$\left\{ \begin{array}{c} \text{Differential string} \\ \text{classes on } M \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Fusion spin structures} \\ \text{on } LM \text{ with superficial} \\ \text{spin connections} \end{array} \right\}_{/\ \text{Iso}}$$

Since we know that differential string classes provide trivializations of the Pfaffian bundle $\mathit{Pfaff}(D)$ over the mapping spaces of all surfaces, this shows that a fusion product is exactly the structure needed to extend Killingback's idea from tori to all surfaces. One can give explicit formulas for these extensions.

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