Functorial field theories and spin geometry

Konrad Waldorf

Universität Greifswald

November 5, 2019

Functorial field theories (FFTs) have been introduced by Atiyah and Segal in order to axiomatize properties of quantum field theories. They have also been used in Khovanov's categorification of the Jones polynomial. I will explain a geometric generalization of FFTs due to Turaev, Stolz, and Teichner, which allows to treat classical and quantum field theories in one setting. The main examples will be Dirac's theory of the electron in an electromagentic field, and the 2d supersymmetric sigma model. The FFT of the latter example requires the so-called stringor bundle on loop space, whose existence on string manifolds has been anticipated by Stolz and Teichner. I will report on recent work joint with Peter Kristel about the construction of this bundle, which involves Fock spaces, von Neumann algebras, and Connes fusion.

Functorial quantum field theories

Smooth functorial field theories

Spin geometry

Central in this talk is not a theorem, but a definition. In fact, this talk is about how to find a well-suited definition, and how to construct relevant examples.

The purpose is to find, on an outmost fundamental level, a mathematical framework to treat field theories, physical theories of particle fields and gauge fields.

Definition (Segal [Seg87], Atiyah [Ati88])

A d-dimensional functorial quantum field theory (FQFT) is a symmetric monoidal functor

$$Z : \mathcal{B}ord_d \to \mathcal{V}ect.$$

$\mathcal{B}ord_d$ is the category of oriented d-dimensional bordisms:

- Objects are $(d-1)$-dimensional closed oriented manifolds Y.
- Morphisms $\Sigma : Y_0 \to Y_1$ are d-dimensional compact oriented manifolds Σ with $\partial\Sigma = \overline{Y_0} \sqcup Y_1$, up to diffeomorphism.
- Composition is defined by gluing along a common boundary.
- The monoidal structure is disjoint union.

$\mathcal{V}ect$ is the category of complex vector spaces, monoidal under the tensor product.

That $Z : \mathcal{B}ord_d \to \mathcal{V}ect$ is a symmetric monoidal functor means:

- It assigns to each $(d-1)$-dimensional closed oriented manifold Y a vector space $Z(Y)$.
- It assigns to each bordism $\Sigma : Y_0 \to Y_1$ a linear map

$$Z(\Sigma) : Z(Y_0) \to Z(Y_1).$$

- The gluing of bordisms corresponds to the composition of linear maps:

$$Z(\Sigma' \cup_{Y_1} \Sigma) = Z(\Sigma') \circ Z(\Sigma).$$

- Disjoint union corresponds to the tensor product:

$$Z(Y \sqcup Y') \cong Z(Y) \otimes Z(Y').$$

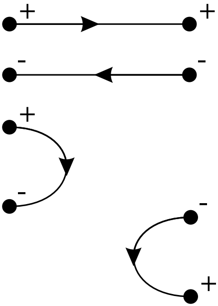
For example, suppose $Z : \mathcal{B}ord_1 \to \mathcal{V}ect$ is a 1-dimensional FQFT.

- There are two objects, each sent to a vector space:

$$\bullet^+ \mapsto V^+$$

$$\bullet^- \mapsto V^-$$

- Morphisms are sent to linear maps:



$$\mapsto \quad \mathrm{id}_{V^+}$$

$$\mapsto \quad \mathrm{id}_{V^-}$$

$$\mapsto \quad d : V^+ \otimes V^- \to \mathbb{C}$$

$$\mapsto \quad b : \mathbb{C} \to V^- \otimes V^+$$

All other objects and morphisms are disjoint unions and compositions of these.

Remarks:

- Two conventions have been fixed und used here:

  End-points of intervals have positive orientation, initial points have negative orientation.

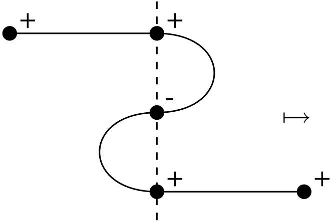
  The orientation on the ingoing points (on the left hand side) is reversed.

- One may interpret $\bullet^{+}$ and $\bullet^{-}$ are particle and anti-particle.

  One may conclude from the interval morphisms that particles move forward (from left to right) and anti-particles move backwards (from right to left).

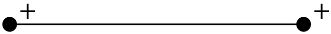
  One may conclude from the curved morphisms that pairs of a particle and an anti-particle can be created and annihilated.

We make two interesting observations. The first is to observe that the following two bordisms are diffeomorphic.



$$\mapsto \quad V^+ \xrightarrow{\mathrm{id}\otimes b} V^+ \otimes V^- \otimes V^+ \xrightarrow{d\otimes \mathrm{id}} V^+$$

and



$$\mapsto \qquad V^+ \xrightarrow{\mathrm{id}} V^+.$$

Hence, they go under the functor Z to *the same* linear maps. We obtain an equality

$$(d \otimes \mathrm{id}) \circ (\mathrm{id} \otimes b) = \mathrm{id}_{V^+}.$$

Looking at negatively oriented intervals, we obtain an analogous equality; together

$$(d \otimes \mathrm{id}) \circ (\mathrm{id} \otimes b) = \mathrm{id}_{V^+}$$
$$(\mathrm{id} \otimes d) \circ (b \otimes \mathrm{id}) = \mathrm{id}_{V^-}$$

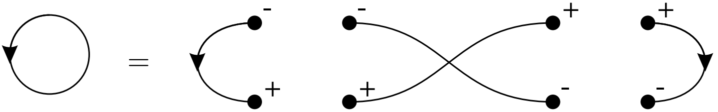
These are algebraic conditions that identify V^+ and V^- as finite-dimensional dual vector spaces:

- d is the "death map", the canonical pairing of a vector space with its dual
- b the "birth map", $b(1) = \sum_{i=1}^{n} b^i \otimes b_i$, for a basis b_i

Theorem (just proved)

1-dimensional FQFTs are the same as finite-dimensional vector spaces.

The second observation is to compute the value of Z on the circle $S^1 : \emptyset \to \emptyset$. We decompose:



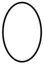
By functoriality, $Z(S^1) : \mathbb{C} \to \mathbb{C}$ is given by:

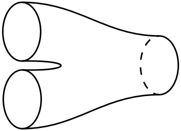
$$\mathbb{C} \xrightarrow{b} V^* \otimes V \xrightarrow{\tau_{V^*,V}} V \otimes V^* \xrightarrow{d} \mathbb{C}$$

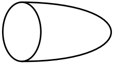
$$1 \longmapsto \sum_{i=1}^{n} b^i \otimes b_i \longmapsto \sum_{i=1}^{n} b_i \otimes b^i \longmapsto \sum_{i=1}^{n} b^i(b_i) = n$$

Thus, the value on the circle gives the only invariant we have: the dimension of the vector space V.

Now we got to dimension two. Suppose $Z : \mathcal{B}ord_2 \to \mathcal{V}ect$ is a 2-dimensional FQFT. All orientation/duality matters are as before, and will now be ignored.

- Objects:  $\mapsto V$
- Morphisms:

 $\mapsto \quad m : V \otimes V \to V$

 $\mapsto \quad tr : V \to \mathbb{C}$

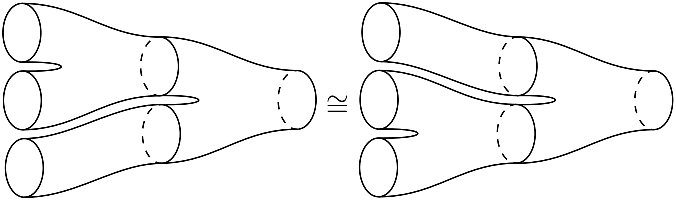
Thus, V is an algebra with a trace. In fact the following is true:

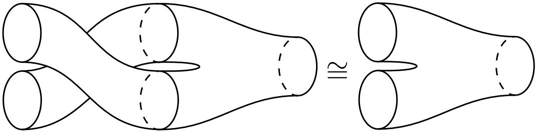
Theorem (Abrams [Abr96], Dijkgraaf [Dij89], Kock [Koc03])

2-dimensional FQFTs are the same as commutative Frobenius algebras.

Remarks:

- Proof of associativity and commutativity: there are orientation-preserving diffeomorphisms





- Value on the torus $S^1 \times S^1$ gives the dimension of the Frobenius algebra.

Motivation to consider FQFTs:

- We want to understand field theories from physics in a fundamental and rigorous way.

  The axioms of a symmetric monoidal functor

  $$Z : \mathcal{B}ord_d \to \mathcal{V}ect$$

  express the *locality* of the theory – a fundamental physical requirement.

- Considering isomorphism classes of objects, an FQFT Z reduces to a *bordism invariant*

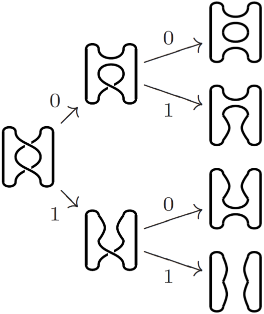
  $$\Omega^{SO}_{d-1} \to \mathbb{Z};$$

  put differently, FQFTs categorify bordism invariants.

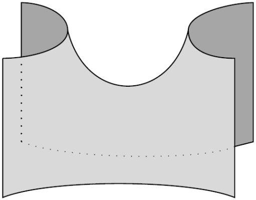
- It has surprising other applications; e.g., Khovanov's link homology.

Khovanov's link homology ("categorified Jones polynomial") for an oriented planar link:

1. resolve crossings in all possible ways:



2. introduce bordisms between the results using the saddle:



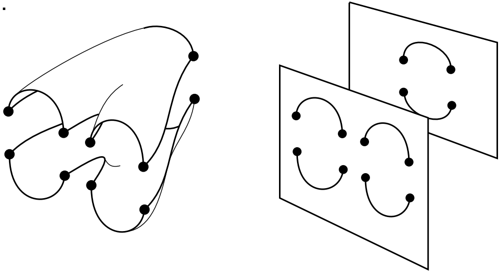
Pictures are taken from [LP09].

3. Choose a Frobenius algebra and apply the associated 2d FQFT; arrange the resulting linear maps in a chain complex.

Rasmussen: proved Milnor conjecture using $k[x]/(x^2 - 1)$.
Kronheimer-Mrowka: Khovanov homology detects unknots.

Remark about extended FQFTs:

- It is interesting to introduce a second "vertical" composition of surfaces:



- This leads to an extension of FQFTs to a higher-categorical setting. The *Baez-Dolan Cobordism Hypothesis* states that extended FQFT are completely determined by its value on the point.

  1d: ✓
  2d: proved by Schommer-Pries [SP09].
  general: proved by Lurie using Joyal's ∞-categories [Lur09].

A little summary:

FQFTs: symmetric monoidal functors $Z : \mathcal{B}ord_d \to \mathcal{V}ect$

- 1d: the same as finite-dimensional vector spaces
- 2d: the same as commutative Frobenius algebras

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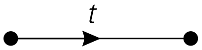
In order to discuss physically relevant theories, we want to add some kind of geometry to the bordisms, for example:

- Metrics, in $d = 1$, to describe Quantum Mechanics:



$$\mapsto \qquad V \xrightarrow{e^{itH}} V$$

- Conformal structures, in order to describe Conformal Field Theory (CFT).
- Spin structures, in order to describe fermions.
- Principal bundles with connections, in $d = 3$, to describe Dijkgraaf-Witten theory or Chern-Simons theory.
- ...

Our main interest for this talk is to equip all manifolds with a smooth map into a fixed manifold X. Usually, we think about this manifold X as the *spacetime* of a classical field theory.

We obtain a new bordism category $\mathcal{B}ord_d(X)$:

- The objects are pairs (Y, ϕ), with $\phi : Y \to X$ a smooth map.
- The morphisms are pairs $(\Sigma, \sigma) : (Y_0, \phi_0) \to (Y_1, \phi_1)$ with $\sigma : \Sigma \to X$ a smooth map such that $\sigma|_{\partial\Sigma} = \phi_0 \sqcup \phi_1$.

Definition

A functorial field theory (FFT) over X is a symmetric monoidal functor

$$Z : \mathcal{B}ord_d(X) \to \mathcal{V}ect$$

Note that "Q" has been dropped!

Warning: the definition of the category $\mathcal{B}ord_d(X)$ is more difficult than it seems here. The main problem is to define the composition of bordisms, in such a way that the map

$$\sigma' \cup \sigma : \Sigma' \cup_{Y_1} \Sigma \to X$$

is again smooth. This is usually done using fixed collars on all bordisms. The main reference are the papers of Stolz-Teichner.

We remark that in $\mathcal{B}ord_d(X)$ to bordisms (Σ, σ) and (Σ', σ') are equivalent if there exists a diffeomorphism $\varphi : \Sigma \to \Sigma'$ that is the identity on the boundary, and satisfies $\sigma = \sigma' \circ \varphi$.

There are versions of $\mathcal{B}ord_d(X)$ where instead a homotopy $\sigma \sim \sigma' \circ \varphi$ is allowed. These are a bit easier to handle, and lead to so-called Homotopy FFTs, as considered by Turaev [Tur10].

Suppose we have an action functional

$$S : C^\infty(\Sigma, X) \to \mathbb{R}$$

for d-dimensional manifolds Σ. For example, when $\Sigma = [a, b]$,

$$S(\sigma) = m \int_a^b \|\mathrm{d}\sigma\|^2 \mathrm{d}t$$

yields – via the principle of the least action – a point-like particle of mass m moving without any forces through the spacetime X.

We get an FFT over X by putting

$$Z(Y, \phi) := \mathbb{C} \quad \text{and} \quad Z(\Sigma, \sigma) = e^{\mathrm{i}S(\sigma)}.$$

Integration rules show that this is a functor:

$$S(const_x) = 0 \quad \text{and} \quad S(\sigma_2 * \sigma_1) = S(\sigma_2) + S(\sigma_1).$$

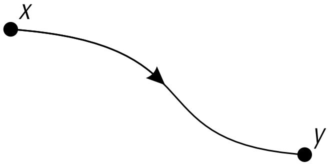
FFTs given by real-valued actions functionals are not so interesting!

Suppose, we have a vector bundle E with connection on X:

- over each point $x \in X$, it has a vector space E_x
- for each arc connecting x with y, it has a linear map

$$pt_E : E_x \to E_y.$$

This gives naturally a 1-dimensional FFT Z_E:



$$\mapsto \qquad E_x \xrightarrow{pt_E} E_y$$

The vector spaces E_x correspond to *state spaces* with "internal degrees of freedom". Since these internal degrees of freedom have to be fixed ("gauged") for local considerations, the FFTs Z_E coming from vector bundles with connection are called *gauge theories*.

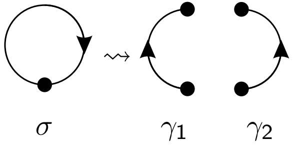
The FFT Z_E describes a point-like particle in the spacetime X that experiences the force of the gauge field E.

The vector bundle is often associated to a principal bundle for a Lie group G via a representation $\rho : G \to \mathrm{GL}(V)$. The standard model of particle physics includes bundles for three Lie groups:

$$U(1) \quad , \quad \mathrm{SU}(2) \quad \text{and} \quad \mathrm{SU}(3).$$

We may compute the value of the FFT Z_E on the circle (S^1, σ). In other words, $\sigma : S^1 \to X$ is a loop, an element $\sigma \in LX$ in the free loop space of X.

We split the loop σ into a left and a right semicircle



$$\sigma = \gamma_1 \cup \gamma_2$$

The associated linear map is

$$\mathbb{C} \xrightarrow{\quad b \quad} E_x^* \otimes E_x \xrightarrow{pt_E(\gamma_1)^{tr-1} \otimes pt_E(\gamma_2)} E_y^* \otimes E_y \xrightarrow{\quad d \quad} \mathbb{C}.$$

It is well known: it is the *trace of the holonomy* of E around σ. Non-trivial holonomy expressed an *Aharonov-Bohm effect* in the setting of FFTs. This effect has been first predicted for electrons in 1949 and measured with satisfying tolerance only around 1985.

Problem: The field theory Z_E "discretizes" the vector bundle E.

Solution: Instead of a single map $\sigma : \Sigma \to X$ to spacetime X we better admit smooth variations

$$\sigma : \Sigma \times T \to X,$$

for arbitrary parameter manifolds T. Doing so on objects and morphisms, we obtain a symmetric monoidal category

$$\mathcal{B}ord_d(X)(T),$$

for each T.

For a smooth map $f : T \to T'$ we obtain a functor

$$\mathcal{B}ord_d(X)(T') \to \mathcal{B}ord_d(X)(T).$$

Topologists call this a *presheaf of symmetric monoidal categories on the category of smooth manifolds*.

This is just like in algebraic geometry. Consider a ring R and the affine scheme $Spec(R)$ represented by R. For each ring T we have a set

$$Spec(R)(T) = Hom(T, R)$$

and for each homomorphism $f : T \to T'$ we have a map

$$Spec(R)(T') \to Spec(R)(T).$$

Affine schemes are *presheaves of sets on the category of rings*.

Here, we look at a presheaves of symmetric monoidal categories on the category of manifolds:

$\mathcal{B}ord_d(X)$	the presheaf of bordisms over X
$\mathcal{VB}un$	the presheaf of vector bundles

Definition (Stolz-Teichner [ST04])

A smooth FFT over X is a morphism

$$Z : \mathcal{B}ord_d(X) \to \mathcal{VB}un$$

of presheaves of symmetric monoidal categories.

For each test manifold T, we get a symmetric monoidal functor

$$Z(T) : \mathcal{B}ord_d(X)(T) \to \mathcal{VB}un(T).$$

What we did before is to only consider $T = pt$.

We upgrade the vector bundle theory Z_E to a *smooth* FFT:

- Objects of $Bord_1(X)(T)$ are smooth maps $\phi : T \to X$.

  We assign to ϕ the pullback $\phi^* E$, which is a vector bundle over T.

- Morphisms $\phi_0 \to \phi_1$ are smooth homotopies

  $$\sigma : [0,1] \times T \to X.$$

  We assign to σ the bundle morphism defined over $t \in T$ by

  $$pt_E(\sigma(-,t)) : E_{\phi(0,t)} \to E_{\phi(1,t)}.$$

Theorem (Freed [Fre95], Schreiber-KW [SW09], ...)

1-dimensional smooth FFTs are the same as vector bundles with connection.

Note: the cobordism hypothesis does not hold for smooth FFTs.

Indeed, only the underlying vector bundle E is involved at the level of objects, not its connection.

We remark that it is not possible to use the presheaf $\mathcal{VB}un^{\nabla}$ of vector bundles *with* connection instead of $\mathcal{VB}un$, as the range of smooth FFTs. Indeed, the bundle morphism

$$Z(T)([0,1],\sigma) : Z(T)(pt,\phi_0) \to Z(T)(pt,\phi_1)$$

is not connection-preserving unless the connection on E is flat.

We try to construct a 2-dimensional smooth FFT

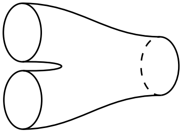
$$Z : \mathcal{B}ord_2(X) \to \mathcal{VB}un.$$

- The objects of $Bord_2(X)(T)$ are pairs (Y, ϕ) with $Y = S^1$ and a smooth map $\phi : T \times S^1 \to X$. We may view this as a smooth map $\Phi : T \to LX := C^\infty(S^1, X)$ into the free loop space of X. Suppose a vector bundle $\mathcal{L}$ over LX is given. Then, we may put

$$Z(T)(Y, \phi) := \Phi^* \mathcal{L}.$$

- The morphisms are of the usual types:

we need a *connection* on $\mathcal{L}$



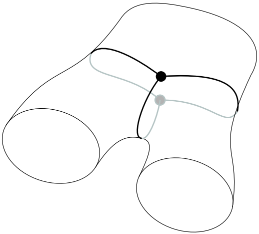
we need a *fusion product* on $\mathcal{L}$

More precisely, a *fusion product* on a vector bundle $\mathcal{L}$ over LX is a family of bundle isomorphisms

$$\lambda : \mathcal{L}_{\gamma_1 \cup \gamma_2} \otimes \mathcal{L}_{\gamma_2 \cup \gamma_3} \to \mathcal{L}_{\gamma_1 \cup \gamma_3}$$

parameterized by triples of paths in X, all connecting the same two points.

Together with the connection and its parallel transport, we obtain a well-defined assignment of linear maps to pairs of pants:



The collection of a line bundle $\mathcal{L}$ over LX with connection and fusion product is closely related to a bundle gerbe with connection over X, and indeed:

Theorem (Bunke-Turner-Willerton [BTW04], KW [Wal16], KW-Bunk [BW])

The following are the same:

1.) *2-dimensional invertible smooth FFTs over X*

2.) *Line bundles over LX with connection and fusion product*

3.) *Bundle gerbes with connection over X*

Remarks

- These smooth FFTs extend to the point, with values in a bicategory of 2-vector bundles
- Bundle gerbes with connection model *B-fields* in string theory. The associated smooth FFT encodes the features of strings moving through the spacetime X in the influence of the B-field.
- Analogous consideration hold for higher dimensional smooth FFTs, with the structure over X coming from higher bundle gerbes with connection.

  This has been investigated in 3d for Chern-Simons theory [Wal13].

For example, the following things correspond under the equivalences of the last theorem:

1.) The WZW model on a Lie group G at level one.

2.) The universal central extension of the loop group LG.

3.) The basic bundle gerbe $\mathcal{G}$ over G.

Most aspects of these relations have been studied in the 80s and 90s by physicists, e.g. Gawędzki [Gaw88].

We observe that smooth FFTs over $X = pt$ are the same as FQFTs. We have classical field theories and quantum field theories in one framework!

The Stolz-Teichner programs aims at identifying smooth FFTs with "objects" of a generalized cohomology theory E, such that we have a bijection

$$d\text{-}sFFT(X) \cong E(X)$$

for all spacetimes X. Using the pushforward map in E, one can define a quantization map

$$\begin{array}{ccc} d\text{-}sFFT(X) & \xrightarrow{\text{Quantization}} & d\text{-}sFFT(pt) = d\text{-}FQFT \\ \| & & \| \\ E(X) & \xrightarrow[!]{} & E(pt) \end{array}$$

This is verified for 1d supersymmetric Euclidean FFTs for which E is K-theory, and the commutativity of above diagram reduces to the Feynman-Kac formula.

A little summary:

FQFTs: symmetric monoidal functors $Z : \mathcal{B}ord_d \to \mathcal{V}ect$

- 1d: finite-dimensional vector spaces
- 2d: commutative Frobenius algebras

Smooth FFTs over X: presheaf morphisms $Z : \mathcal{B}ord_d(X) \to \mathcal{VB}un$

- 1d: vector bundles with connection over X
- 2d, invertible: line bundles with connection and fusion product over LX

Functorial quantum field theories

Smooth functorial field theories

Spin geometry

Joint work with Peter Kristel and Matthias Ludewig.

We start in dimension one, and recall the classical treatment of a (massless) fermion on the circle S^1. We require a spin structure with associated spinor bundle $\mathbb{S}$ on S^1.

Let $\sigma : S^1 \to X$ describe the position of the fermion in spacetime. We consider the vector bundle $\mathbb{S}_\sigma := \mathbb{S} \otimes \sigma^* TX$ over S^1, and the Hilbert space $H_\sigma := L^2(\mathbb{S}_\sigma)$ of square-integrable sections ψ, which constitute the fermions. On H_σ acts a twisted Dirac operator

$$D_\sigma : H_\sigma \to H_\sigma.$$

The action functional for a fermion ψ is

$$S_\sigma(\psi) := \langle \psi, D_\sigma \psi \rangle \,,$$

and the classical equation of motion is the Dirac equation,

$$D_\sigma \psi = 0.$$

In order to get into the context of smooth FFTs, we want to perform the fermionic path integral

$$\mathcal{A}(\sigma) := \int_{\psi \in L^2(\mathbb{S}\otimes\sigma^* TX)} e^{\langle \psi, D\psi \rangle} d\psi.$$

A rigorous interpretation and evaluation of this path integral requires a spin structure on X. Then, $\mathcal{A}$ becomes a smooth map

$$\mathcal{A} : LX \to \mathbb{C}.$$

It is a square-root of the zeta-regularized determinant of the covariant derivative; these are results of Atiyah [Ati85], Freed [Fre86], and Prat-Waldron [PW09].

Can we extend this to a 1-dimensional smooth FFT over X?

Equivalently, is there a vector bundle E with connection over X such that $\mathrm{tr}(\mathrm{Hol}_E) = \mathcal{A}$?

Yes, it is the spinor bundle on X, equipped with its natural connection. This was proved by Prat-Waldron [PW09].

Remarks:

- Note that the extension to a smooth FFT is much simpler to describe than its value on the circle!
- The family of Dirac operators D_σ defines a Pfaffian line bundle Pf over LX. Before involving a spin structure on X, the fermionic path integral is a section $\mathcal{A} : LX \to Pf$; the theory has a *fermionic anomaly*. A spin structure on X induces a trivialization of Pf, and hence renders $\mathcal{A}$ a complex-valued function.
- We have not mentioned that the spinor bundle is a module for a bundle of Clifford algebras, and that parallel transport in the spinor bundle is Clifford-linear. This can be ignored here, but will be essential in two dimensions.

Finally, we look at dimension two.

Analogously, for Σ a closed surface, the fermionic path integral is a section in a Pfaffian line bundle over $C^\infty(\Sigma, X)$. This bundle is trivialized by a (geometric) *string structure* on X, using results of Bunke [Bun11]. For the corresponding function

$$\mathcal{A} : C^\infty(\Sigma, X) \to \mathbb{C}$$

no alternative description is known.

Can we extend this to a 2-dimensional smooth FFT over X?

We need vector bundle over LX with connection and fusion product, the *spinor bundle on loop space*. This bundle has been attempted to construct in "formal" ways or special cases by Witten [Wit86], Brylinski [BM92], Taubes, and others. Major progress was achieved by Stolz and Teichner [ST] using string structures, yet, their construction remained unfinished.

We sketch the main step in a rigorous construction, obtained in collaboration with Peter Kristel [KWb] and [KWa].

First we work over a fixed loop, $\phi : S^1 \to X$. Consider the real Hilbert space

$$H_\phi := L^2(\mathbb{S} \otimes \phi^* TX).$$

Its Clifford algebra $Cl(H_\phi)$ acts on the Fock space

$$\mathcal{F}_L := \Lambda L$$

where L is a Lagrangian subspace of H_ϕ. Problem: L cannot be chosen such that it varies continuously over LX: the Fock spaces $\mathcal{F}_L$ do not combine into a continuous bundle.

The modelling Hilbert space $H_0 := L^2(\mathbb{S} \otimes \mathbb{C}^n)$ has a distinguished Lagrangian L_0, the so-called *Atiyah-Patodi-Singer Lagrangian* of "spinors that extend to anti-holomorphic functions on the disk".

Consider a local frame at $\phi \in LX$, inducing an isometric isomorphism $\varphi : H_\phi \cong H_0$. Then, $L_\varphi := \varphi^{-1} L_0$ is a Lagrangian in H_ϕ. Another frame φ' is related to φ by a transformation $g \in LO(n)$. We need to correct the difference by letting g act on the Fock space $\mathcal{F}_0 = \mathcal{F}_{L_0}$.

But $LO(n)$ does not act on $\mathcal{F}_0$.

If X is oriented, we may assume $g \in LSO(n)$. But $LSO(n)$ does not act on $\mathcal{F}_0$, either.

If X is a spin manifold, we may assume $g \in LSpin(n)$. But $LSpin(n)$ still does not act on $\mathcal{F}_0$.

If X is a string manifold, then the frame bundle of LX is lifted along universal central extension of Pressley-Segal [PS86],

$$U(1) \to \widetilde{LSpin}(n) \to LSpin(n).$$

It maps into another central extension studied by Araki [Ara74],

$$U(1) \longrightarrow Imp(H_0) \longrightarrow O_{res}(H_0),$$

where $Imp(H_0)$ is the *group of implementers*, which acts on $\mathcal{F}_0$.

Lemma

If X is a string manifold, this construction yields a smooth bundle $\mathcal{F}$ of Fock spaces over LX.

Similarly, a *geometric* string structure on X induces a connection on $\mathcal{F}$.

The group $LSpin(n)$ acts on the Clifford C*-algebra $Cl(H_0)$ by *Bogoliubov automorphisms*, so that one can define a smooth bundle Cl of C*-algebras, as soon as X is a spin manifold. This bundle acts irreducibly on the spinor bundle $\mathcal{F}$.

We prove that there exists a bundle A of C*-algebras over the path space PX, such that

$$Cl_{\gamma_1 \cup \gamma_2} \cong A_{\gamma_1} \cup A_{\gamma_2}^{op}$$

for the completion to bundles of von Neumann algebras.

Lemma

Over a loop $\gamma_1 \cup \gamma_2 \in LX$, the Fock space $\mathcal{F}_{\gamma_1 \cup \gamma_2}$ is a an irreducible von Neumann $\mathcal{A}_{\gamma_1}$-$\mathcal{A}_{\gamma_2}$-bimodule.

Our main result is to define a version of a fusion product on the spinor bundle $\mathcal{F}$.

Theorem (KW-Kristel [KWa])

The spinor bundle $\mathcal{F}$ on loop space is equipped with a fusion product

$$\mathcal{F}_{\gamma_1 \cup \gamma_2} \boxtimes_{\mathcal{A}_{\gamma_2}} \mathcal{F}_{\gamma_2 \cup \gamma_3} \cong \mathcal{F}_{\gamma_1 \cup \gamma_3}.$$

Here, $\boxtimes_{\mathcal{A}_{\gamma_2}}$ denotes the Connes fusion tensor product of bimodules over von Neumann algebras.

With this fusion product, we have constructed an infinite-dimensional vector bundle $\mathcal{F}$ over LX with connection and fusion product.

These are the building blocks for casting the free fermion in dimension two as a smooth FFT! (The full picture is subject of going work with Peter Kristel and Matthias Ludewig.)

Some open questions:

- The construction and properties of a *Dirac operator* on loop space, acting on sections of $\mathcal{F}$. The construction of this operator is a long outstanding open problem.
- The relation between spin geometry on loop spaces and *elliptic cohomology*, which should be similar to the relation between ordinary spin geometry and K-theory.
- The development of an *index theory on loop space*. The expectation is such this would provide a proof of the Stolz conjecture about positive Ricci curvature.

A little summary:

FQFTs: symmetric monoidal functors $Z : \mathcal{B}ord_d \to \mathcal{V}ect$

- 1d: finite-dimensional vector spaces
- 2d: commutative Frobenius algebras

Smooth FFTs over X: presheaf morphisms $Z : \mathcal{B}ord_d(X) \to \mathcal{VB}un$

- 1d: vector bundles with connection over X
- 2d, invertible: line bundles with connection and fusion product over LX

Examples in spin geometry:

- 1d: the spinor bundle on a spin manifold X
- 2d: the spinor bundle on the loop space LX of a string manifold

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