Transgression of higher structures to loop spaces

Workshop on Loop space and Higher category

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$$
\begin{tikzcd}
 & \text{String theory} & \\
\text{B-Fields} \arrow[ur] & & \text{Configuration space} \arrow[ul]
\end{tikzcd}
$$

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\text{B-Fields} \arrow[ur] & & \text{Configuration space} \arrow[ul] \\
\text{Higher-categorical perspective} \arrow[rr, leftrightarrow] & & \text{Loop-space perspective}
\end{tikzcd}
$$

Configuration space $LM = C^\infty(S^1, M)$.

"a point in LM is a string in M"

View string theory on M as a theory for point-particles in $X = LM$.

We know many things about point-particles:

- gauge fields
- spin structures

Look at **gauge fields** more closely ($X = LM$).

- Field strength $F \in \Omega^2(X)$
- Homogeneous Maxwell equations $\mathrm{d}F = 0$.
- Gauge potential $A \in \Omega^1(X)$, $\mathrm{d}A = F$.
- Coupling term

$$S(\gamma, A) = \int_0^1 \gamma^* A$$

  for $\gamma : [0, 1] \to X$ trajectory.

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for $\gamma : [0,1] \to X$ trajectory.

Problem: $[F] \in \mathrm{H}^2_{dR}(X)$ may be non-trivial!

"Magnetic monopole"

Solution (Dirac 1931):

- local gauge fields $A_\alpha \in \Omega^1(U_\alpha)$, $\mathrm{d}A_\alpha = F$
- gauge transformation on the overlaps:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(1)$$

  such that

$$A_\beta = A_\alpha + \mathrm{d}\log(g_{\alpha\beta})$$

  and

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}.$$

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- well-defined coupling

$$\prod_{i=1}^{n} \exp\left(2\pi\mathrm{i}\int_{t_{i-1}}^{t_i} \gamma^* A_{\alpha_i}\right) g_{\alpha_i\alpha_{i+1}}(\gamma(t_i)) \in \mathrm{U}(1).$$

In modern terminology of Differential Geometry, the collection

$$(A_\alpha, g_{\alpha\beta})$$

represents a

principal $\mathrm{U}(1)$-bundle P with connection over X.

- The field strength F is the **curvature** of P.
- The coupling term is the **parallel transport** of P along the path γ.

  It becomes a map between the fibres:

  $$pt_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$$

$\boxed{\text{Fact 1}}$ Principal $\mathrm{U}(1)$-bundles P with curvature F exists if and only if F has **integral periods**: the class $[F] \in \mathrm{H}^2_{dR}(X)$ is in the image of

$$\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{R}) = \mathrm{H}^2_{dR}(X).$$

"charge quantization"

$\boxed{\text{Fact 2}}$ Possible choices of P for fixed curvature F are parameterized by $\mathrm{H}^1(X, \mathrm{U}(1))$.

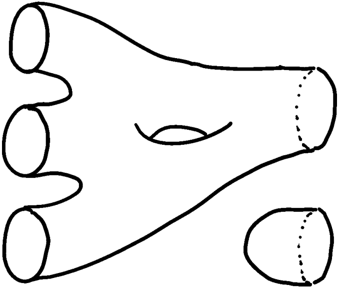
"Aharonov-Bohm effect"

Summary:

The configuration space perspective suggests that we should consider a principal $\mathrm{U}(1)$*-bundle with connection over the loop space* $X = LM$*.*

Problem: a curve in $X = LM$ is just a **cylinder** in M:

In order to describe string theory, we need to consider **more general shapes**, e.g.



A general **string trajectory** is a compact surface

$$\phi : \Sigma \to M.$$

Let us assume first that Σ is **oriented** and has **no boundary**.

If $B \in \Omega^2(M)$, then we could define

$$\int_\Sigma \phi^* B$$

as the coupling of the string to the "B-Field" B.

In this context, $H := \mathrm{d}B \in \Omega^3(M)$, is called the **H-flux**.

Important example: the **Wess-Zumino-Witten model**.

(Wess-Zumino 1971, Witten 1984, Novikov 1981)

Here: $M = G$ is a Lie group, e.g. $G = \mathrm{SU}(2)$.

The H-flux is the **Cartan 3-form**,

$$H := \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G).$$

In many interesting cases, $[H] \in \mathrm{H}^3_{dR}(G)$ is non-trivial: no global B-field exists.

Copy the Dirac method (Alvarez 1984, Gawedzki 1988):

- local B-fields $B_\alpha \in \Omega^2(U_\alpha)$, $\mathrm{d}B = H|_{U_\alpha}$
- gauge potentials $B_\beta = B_\alpha + \mathrm{d}A_{\alpha\beta}$
- gauge transformations

  $$A_{\alpha\gamma} = A_{\alpha\beta} + A_{\beta\gamma} + \mathrm{dlog}(g_{\alpha\beta\gamma})$$

  such that

  $$g_{\alpha\beta\gamma} \cdot g_{\alpha\gamma\delta} = g_{\alpha\beta\delta} \cdot g_{\beta\gamma\delta}.$$

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Well-defined coupling: triangulate the surface Σ, with faces f, edges e, and vertices v:

$$\prod_f \exp\left(2\pi\mathrm{i}\int_f \phi^* B_{\alpha_f}\right) \prod_{e\in\partial f} \exp\left(2\pi\mathrm{i}\int_e \phi^* A_{\alpha_f\alpha_e}\right) \prod_{v\in\partial e} g^{\epsilon(f,e,v)}_{\alpha_f\alpha_e\alpha_v}(v).$$

In modern terminology, the data $\mathcal{G} = (B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma})$ represents a

bundle gerbe $\mathcal{G}$ with connection over M.

(Murray 1995, Murray-Stevenson 2000, Carey et al. 2003)

- The H-flux H is the **curvature** of $\mathcal{G}$.
- The coupling term is the **surface holonomy** of $\mathcal{G}$ around the surface $\phi : \Sigma \to M$.

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$\boxed{\text{Fact 1}}$ A bundle gerbe $\mathcal{G}$ with curvature H exists if and only if H has integral periods.

"Flux quantization"

$\boxed{\text{Fact 2}}$ Possible choices parameterized by $\mathrm{H}^2(G, \mathrm{U}(1))$.

"Discrete torsion"

Consider the **Wess-Zumino-Witten model** on a Lie group of Cartan type (compact + simple + connected + simply connected).

Then, $\mathrm{H}^3(G, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{H}^2(G, \mathrm{U}(1)) = 0$.

Moreover, $[H] = 1$.

Thus, for each "level" $k \in \mathbb{Z}$, there is a unique bundle gerbe $\mathcal{G}_k$ over G with connection of curvature kH.

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The **basic gerbe** $\mathcal{G}_1$ has a concrete Lie-theoretical construction (Meinrenken 2002, Gawędzki-Reis 2002)

- U_α conjugation-invariant, $\alpha = 0, ..., \mathrm{rk}(\mathfrak{g})$
- $U_\alpha \cap U_\beta$ retracts to a coadjoint orbit $\mathcal{O}_{\lambda_\alpha - \lambda_\beta} \subseteq \mathfrak{g}^*$, for λ_α vertices of Weyl alcove
- $P_{\alpha\beta}$ is the prequantum bundle with its Kostant connection

A bundle gerbe with connection is a **higher structure**: they form a symmetric monoidal bicategory

$$\mathcal{G}rb^{\nabla}(M).$$

E.g.,

$$\mathrm{Hom}_{\mathcal{G}rb^{\nabla}(M)}(\mathcal{G},\mathcal{G}) \cong \mathcal{B}un^{\nabla_0}_{\mathrm{U}(1)}(M).$$

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$\boxed{\text{Fact}}$ (Stevenson 2000) The assignment

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is a **sheaf of bicategories** on the site of smooth manifolds.

However, $M \mapsto \mathrm{h}_0\mathcal{G}rb^{\nabla}(M)$ is **not** a sheaf of categories.

This is in fact **relevant** in **Wess-Zumino-Witten models**.

If G is a compact simple Lie group, then $G = \tilde{G}/Z$, with

- $\tilde{G}$ of Cartan type
- $Z \subseteq Z(\tilde{G})$

In order to construct bundle gerbes with connection over G, we let them descend from $\tilde{G}$.

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Gawedzki-Reis classified all Z-**equivariant structures** on the bundles gerbes $\mathcal{G}_k$, for all $\tilde{G}$ and all Z.

Example 1: $\tilde{G} = \mathrm{SU}(2)$ and $Z = \mathbb{Z}_2$. Then, $\mathcal{G}_k$ admits Z-equivariant structures if and only if k is even, and in this case, there is exactly one. Thus, there is precisely one bundle gerbe over $\mathrm{SO}(3)$ for each even level k.

Example 2: $\tilde{G} = \mathrm{Spin}(4)$ and $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$. Again, k must be even but then $\mathcal{G}_k$ admits two different Z-equivariant structures. Thus, there are two bundle gerbes over $\mathrm{PSO}(4)$ for each even level.

Using bundle gerbes and their bicategorical structure, one can get rid of the assumption that Σ is **oriented**

"Jandl structure"

(Schreiber-Schweigert-KW 2005, Gawedzki-Suszek-KW 2007)

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A further aspect that could successfully be treated using bundle gerbes are open strings, whose end points are constrained to **D-branes**.

(Kapustin 2000, Gawedzki-Reis 2002, Gawedzki 2004, Carey et al. 2005)

Summary:

The B-field perspective suggests to use bundle gerbes with connections in order to describe gauge fields for strings. That way, it succeeded to describe the coupling to strings for all compact surfaces.

We want to get rid of the assumption that Σ has **no boundary**.

$\boxed{\text{Fact}}$ (Brylinski 1993) Surface holonomy remains well-defined for surfaces with boundary if a trivialization of $\mathcal{G}$ over $\partial\Sigma$ is fixed.

Let:

- $b_1, \ldots, b_n \subseteq \partial\Sigma$ label "incoming" boundary components
- $c_1, \ldots, c_m \subseteq \partial\Sigma$ label "outgoing" boundary components
- $S^1 \overset{\varphi_i}{\cong} b_i$ orientation-reversing diffeomorphisms
- $S^1 \overset{\psi_i}{\cong} c_i$ orientation-preserving diffeomorphisms

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Denote, for a loop $\tau \in LM$, by

$$L\mathcal{G}_\tau := \{ \text{ Trivializations of } \tau^*\mathcal{G} \} \ / \ \text{2-isomorphism}$$

Then, **surface holonomy** is a well-defined map

$$L\mathcal{G}_{\phi\circ\varphi_1} \otimes \dots \otimes L\mathcal{G}_{\phi\circ\varphi_n} \to L\mathcal{G}_{\phi\circ\psi_1} \otimes \dots \otimes L\mathcal{G}_{\phi\circ\psi_m}$$

$\boxed{\text{Fact 1}}$ $L\mathcal{G}_\tau$ is a $\mathrm{U}(1)$-torsor:

$$\mathrm{h}_0\mathrm{Hom}_{\mathcal{G}rb^\nabla(S^1)}(\mathcal{I},\mathcal{I}) \cong \mathrm{h}_0\mathcal{B}un^{\nabla_0}_{\mathrm{U}(1)}(S^1) \cong \mathrm{U}(1).$$

$\boxed{\text{Fact 2}}$ $L\mathcal{G}$ is a principal $\mathrm{U}(1)$-bundle over LM.

$\boxed{\text{Fact 3}}$ There exists a unique connection on $L\mathcal{G}$ whose parallel transport along a path τ_t in LM is the surface holonomy map

$$L\mathcal{G}_{\tau_0} \longrightarrow L\mathcal{G}_{\tau_1}$$

Its curvature is

$$F = \int_{S^1} \mathrm{ev}^* H$$

The assignment

$$\mathcal{G} \mapsto L\mathcal{G}$$

extends to a functor

$$\mathrm{h}_1\mathcal{G}rb^{\nabla}(M) \to \mathcal{B}un^{\nabla}_{\mathrm{U}(1)}(LM)$$

This functor is called **transgression** – it unifies our two perspectives to string theory in M.

Connections in the image of transgression are particular.

$\boxed{\text{Fact 1}}$ Surface holonomy for closed Σ is trivial when $\phi : \Sigma \to M$ is thin, i.e., $\mathrm{d}\phi_x$ is a rank-one-map.

Thus, **thin loops** in LM have trivial holonomy.

Equivalently, the parallel transport along a thin path does not depend on the path.

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$\boxed{\text{Fact 2}}$ Surface holonomy is invariant under thin homotopies, i.e., if $\phi_0, \phi_1 : \Sigma \to M$ are homotopic via a homotopy $h_t : \Sigma \to M$ such that $h : [0,1] \times \Sigma \to M$ has rank two, then ϕ_0 and ϕ_1 have the same surface holonomy.

Thus, **rank-two-homotopic loops** in LM have equal holonomy.

A connection on a principal $\mathrm{U}(1)$-bundle on LM is called **superficial** if it has both properties.

Superficial connections render a principal bundle **equivariant** for the action of $\mathcal{D}\mathrm{iff}^+(S^1)$ on LM.

Recall that $\mathcal{D}\mathrm{iff}^+(S^1)$ is connected: for every element $\phi \in \mathcal{D}\mathrm{iff}^+(S^1)$ there is a path ϕ_t with $\phi_0 = \mathrm{id}_{S^1}$ and $\phi_1 = 1$.

Consider a loop $\tau \in LM$. Then, $\tau \circ \phi_t$ is a path in LM from τ to $\tau \circ \phi$. Parallel transport gives a map

$$L\mathcal{G}_\tau \to L\mathcal{G}_{\tau\circ\phi},$$

lifting the action of $\mathcal{D}\mathrm{iff}^+(S^1)$ from LM to $L\mathcal{G}$.

This does not depend on the choice of ϕ_t, because $\tau \circ \phi_t$ is thin:

$$[0,1] \times S^1 \xrightarrow{\phi} S^1 \xrightarrow{\tau} M$$

factors through the 1-dimensional manifold S^1.

A bundle gerbe with connection $\mathcal{G}$ over M describes string couplings for **all compact oriented surfaces** Σ.

The principal $\mathrm{U}(1)$-bundle $L\mathcal{G}$ can only describe couplings for **cylinders**.

Thus, **transgression looses information**.

How can this information be recovered?

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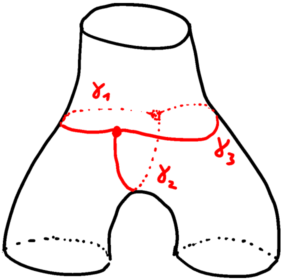
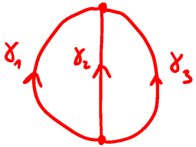
Idea: a general surface has a pair-of-pants-decomposition, i.e., it can be chopped up into pairs-of-pants, cylinders, and caps.

We have to take care about the pairs-of-pants.

A pair-of-pants, in turn, consists of three cylinders and some "product"



$\rightsquigarrow$



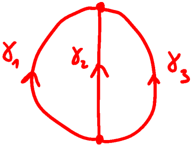
If two nice paths γ and γ' have a common initial point and a common end point, then they form a loop $\gamma \cup \gamma' := \overline{\gamma'} * \gamma$.

The three cylinders end at the loops $\gamma_1 \cup \gamma_2$, $\gamma_2 \cup \gamma_3$, and $\gamma_1 \cup \gamma_3$.

A **fusion product** on a principal $\mathrm{U}(1)$-bundle P over LM is a family of bundle morphisms

$$\lambda_{\gamma_1,\gamma_2,\gamma_3} : P_{\gamma_1 \cup \gamma_2} \otimes P_{\gamma_2 \cup \gamma_3} \to P_{\gamma_1 \cup \gamma_3}$$

for each triple of paths



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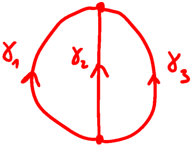

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$L\mathcal{G}$ has a fusion product, given by surface holonomy

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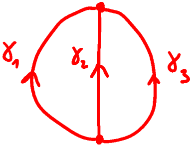

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$\boxed{\text{Fact}}$ (KW 2010) Transgression is an **equivalence of categories**:

$$\mathcal{G}rb^{\nabla}(M) \cong \mathcal{F}us\mathcal{B}un^{\nabla^{sf}}_{\mathrm{U}(1)}(LM).$$

Summary:

Transgression establishes an equivalence between the two perspectives described before, after adding structure (fusion product) and conditions (superficiality) on the loop space side.

We may use transgression in order to recast string theory in the framework of **smooth functorial field theories**:

$$Z : \mathrm{Bord}_2^{\mathrm{or}}(M) \to \mathrm{VectBdl}_{\mathbb{C}}$$

Here:

- $\mathrm{Bord}_2^{\mathrm{or}}(M)$ is a sheaf of categories over the site of manifolds.

  It assigns to a "test" manifold T the category whose objects are closed oriented 1-dimensional manifolds S together with a T-family of smooth maps to M, i.e., $\phi : T \times S \to M$.

  The morphisms are compact oriented 2-dimensional bordisms Σ together with a T-family of smooth maps to M, modulo diffeomorphism.

- $\mathrm{VectBdl}_{\mathbb{C}}$ is the sheaf of complex vector bundles.
- Z is a morphism of symmetric monoidal stacks.

Smooth functorial field theories have been invented by Stolz and Teichner in order control smooth families of bordisms.

Evaluating at a 1-point manifold $T = *$ reproduces the notion of an ordinary functorial field theory (or TQFT), where the bordisms just have a fixed map to M.

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Smooth functorial field theories can have additional features:

- invertible: take only values in complex **line** bundles.
- reflection-positive: exchange a flip operation on the bordisms with complex conjugation of vector spaces. (Freed-Hopkins 2017)
- superficial: the value on morphisms depends only on the thin homotopy class of the map to M.

Given a bundle gerbe with connection over M, we want to define a **functorial field theory** $Z_{\mathcal{G}}$ with

$$Z_{\mathcal{G}}(T)(S^1, \phi) := (\phi^\vee)^* L\mathcal{G},$$

where $\phi^\vee \in C^\infty(T, LM) \leftrightarrow C^\infty(T \times S^1, M) \ni \phi$.

$\boxed{\text{Fact}}$ The surface holonomy of $\mathcal{G}$ has all required properties to make this a well-defined invertible, reflection-positive, superficial smooth functorial field theory.

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$$\begin{tikzcd}
\left\{ \begin{array}{c} \text{Bundle gerbes with} \\ \text{connection over } M \end{array} \right\} \arrow[rr, leftrightarrow, "\text{Transgression}"] & & \left\{ \begin{array}{c} \text{Fusive principal} \\ \mathrm{U}(1)\text{-bundles over} \\ LM \text{ with superficial} \\ \text{connections} \end{array} \right\} \\
& \left\{ \begin{array}{c} \text{2-dimensional invertible} \\ \text{reflection-positive superficial} \\ \text{smooth functorial field} \\ \text{theories on } M \end{array} \right\} \arrow[ul, leftrightarrow, "\text{Bunke-Turner-Willerton 2004}"] \arrow[ur, leftrightarrow, "\text{Bunk-KW 2018}"'] &
\end{tikzcd}$$

Gauge theory on loop space

Bundle gerbes

Transgression

Smooth functorial field theory

Multiplicative bundle gerbes

Spin structures on loop space

Summary

Bundle gerbes over Lie groups can be **multiplicative**:

- 1-morphism
$$\mathcal{M} : \mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G} \to m^*\mathcal{G}$$

- 2-isomorphism
$$\begin{tikzcd}
\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{G}_3 \arrow[r, "\mathcal{M}_{1,2}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathcal{M}_{2,3}"'] & \mathcal{G}_{12} \otimes \mathcal{G}_3 \arrow[d, "\mathcal{M}_{12,3}"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{G}_1 \otimes \mathcal{G}_{23} \arrow[r, "\mathcal{M}_{1,23}"'] & \mathcal{G}_{123}
\end{tikzcd}$$

  satisfying a pentagon axiom.

(Carey et al. 2003)

Since transgression is a **functor**, we obtain a bundle morphism

$$\mathrm{pr}_1^* L\mathcal{G} \otimes \mathrm{pr}_2^* L\mathcal{G} \to m^* L\mathcal{G}$$

This is the same as a group structure on $L\mathcal{G}$ turning it into a **central extension**

$$1 \to \mathrm{U}(1) \to L\mathcal{G} \to LG \to 1.$$

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$\boxed{\text{Fact}}$ Let G be a Lie group of Cartan type. Each bundle gerbe $\mathcal{G}_k$ admits exactly one multiplicative structure. The transgression of the basic gerbe $\mathcal{G}_1$ yields the **universal central extension**

$$1 \to \mathrm{U}(1) \to \widetilde{LG} \to LG \to 1.$$

(Pressley-Segal 1986, KW 2007)

Additionally, we obtain:

- A **superficial** connection.

  Note that a connection on a central extension determines a splitting of the Lie algebra extension.

- In particular, a $\mathcal{D}\mathrm{iff}^+(S^1)$-equivariant structure.
- A multiplicative **fusion product**.

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This additional structure can be used to distinguish "transgressive" central extensions from arbitrary ones (KW 2016).

For example, every central extension of the loop group of a compact simple Lie group is transgressive.

$L\mathrm{U}(1)$ has non-transgressive central extensions.

Supersymmetric field theories suffer from a "global anomaly"

- 1-dimensions: anomaly represented by 2nd Stiefel-Whitney class
$$w_2 \in \mathrm{H}^2(M, \mathbb{Z}_2)$$
Cancellation: spin structure on M
- 2-dimensions: anomaly represented by
$$\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$$
Cancellation: again two approaches:

  1.) Killingback 1987: spin structure on LM

  2.) Stolz-Teichner 2004: **string structure** on M

Spin structures on loop spaces (Killingback 1987):

- M a spin manifold of dimension n

  $\rightsquigarrow$ frame bundle FM is a $\mathrm{Spin}(n)$-principal bundle

  $\rightsquigarrow$ looped bundle LFM is a $L\mathrm{Spin}(n)$-principal bundle

- Definition: a **spin structure** on LM is a lift of the structure group of LFM to the universal central extension

  $$1 \to \mathrm{U}(1) \to \widetilde{L\mathrm{Spin}(n)} \to L\mathrm{Spin}(n) \to 1$$

  i.e. a principal $\widetilde{L\mathrm{Spin}(n)}$-bundle $\widetilde{LFM}$ over LM with an equivariant map $\sigma : \widetilde{LFM} \to LFM$.

Obstruction against spin structures on loop spaces:

- Spin structures exists if and only if a certain class
$$\lambda_{LM} \in \mathrm{H}^3(LM, \mathbb{Z})$$
vanishes.
- Theorem [McLaughlin '92]:
$$\lambda_{LM} = \int_{S^1} \mathrm{ev}^* \left(\frac{1}{2} p_1(M) \right)$$
- Thus, we have
$$\frac{1}{2} p_1(M) = 0 \quad \Rightarrow \quad \lambda_{LM} = 0$$
but the converse is not true in general (Pilch-Warner 1988)

$\rightsquigarrow$ we need enhanced notion of spin structures on loop spaces

General lifting theory provides a **reformulation** in terms of principal S^1-bundles and bundle isomorphisms:

- The equivariant map

$$\sigma : \widetilde{LFM} \to LFM$$

  exhibits $\widetilde{LFM}$ as a principal $\mathrm{U}(1)$-bundle over LFM.

General lifting theory provides a **reformulation** in terms of principal S^1-bundles and bundle isomorphisms:

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exhibits $\widetilde{LFM}$ as a principal $\mathrm{U}(1)$-bundle over LFM.

- The principal $\widetilde{L\mathrm{Spin}(n)}$-action on $\widetilde{LFM}$ can be encoded as an isomorphism

$$\kappa : \widetilde{LFM} \otimes \widetilde{L\mathrm{Spin}(n)} \to \rho^* \widetilde{LFM},$$

of $\mathrm{U}(1)$-bundles over $LFM \times L\mathrm{Spin}(n)$, with ρ the principal action of $L\mathrm{Spin}(n)$ on LFM.

Enhanced version of a spin structure:

- Definition: A **fusive spin structure** on LM is a spin structure $\widetilde{LFM}$ with a fusion product λ on its $\mathrm{U}(1)$-bundle over LFM such that
$$\kappa : \widetilde{LFM} \otimes \widetilde{L\mathrm{Spin}(n)} \to \rho^* \widetilde{LFM}$$
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- $\boxed{\text{Fact 1}}$ (KW 2012) Fusion spin structures exist if and only if
$$\frac{1}{2}p_1(M) = 0$$
- $\boxed{\text{Fact 2}}$ (KW 2016) Transgression establishes an equivalence of bicategories
$$\left\{ \begin{array}{c} \text{Geometric string} \\ \text{structures on } M \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Superficially geometric} \\ \text{fusive spin structures on } LM \end{array} \right\}$$

$\rightsquigarrow$ more in the talks of Peter Kristel and Matthias Ludewig.

Summary:

- Bundle gerbes with connection are a differential geometric **higher structure** on a manifold M.
- Transgression establishes an equivalence to ordinary geometric structure on the **loop space** LM, fusive principal bundles with superficial connections.
- Induced/refined equivalences exist between:
  (a) Multiplicative bundle gerbes on G and certain central extensions of LG.
  (b) String structures on M and fusive spin structures on LM.

Thank you very much for your attention!