Introduction to Gerbes in Conformal Field Theory

Part I: Closed strings

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The closed Wess-Zumino-Witten-model. The Wess-Zumino-Witten-model is a two-dimensional, non-linear sigma model, whose target space is a Lie group G. That is a theory of maps $g \in C^{\infty}(\Sigma, G)$ from a closed oriented worldsheet Σ into G, where G will also be compact in this talk.

It can be obtained as a loop space analogon of the motion of a quantum particle on G. Classicaly, this particle has a $G \times G$-symmetry by left and right translation. This symmetry is still present in the quantized theory, where the Hilbert space $\mathcal{H} = L^2(G, \mathrm{d}g)$ decomposes ($\rightarrow$ Peter-Weyl theory) in a direct sum

$$\mathcal{H} = \bigoplus_{R \text{ irred.}} V_R \otimes V_{\overline{R}}.$$

A two-dimensional generalization of the action of the quantum particle is the following kinetic term

$$S_{\mathsf{kin}}(g) := \frac{k}{2} \int_{\Sigma} \langle g^*\theta \wedge \star g^*\theta \rangle$$

with a coupling contant k. However, the quantization of a model with this action leads to a theory without conformal invariance.

To restore conformal invariance of the quantized theory, one adds the Wess-Zumino-term

$$S_{\mathsf{WZ}}(g) := k \int_B \widetilde{g}^* H,$$

where $H = \frac{1}{12\pi} \langle \theta, [\theta, \theta] \rangle$ is a closed biinvariant threeform, B is a three-dimensional manifold with boundary $\partial B = \Sigma$, and $\widetilde{g}$ is an extension of g to B. The existence and well-definedness of this term is not clear. Witten restricted himself to connected, simple and simply connected Lie groups. Then $\pi_2(G) = 0$, and the two-connectedness of G garantuees the existence of the extension $\widetilde{g}$. Furthermore, H defines an integral class $[H] \in \mathrm{H}^3(G, \mathbb{Z})$ in cohomology. Therefore the ambiguities of the Wess-Zumino-Term, due to different choices of B or $\widetilde{g}$, take values in $k\mathbb{Z}$.

Then the action

$$S(g) := S_{\mathsf{kin}}(g) + S_{\mathsf{WZ}}(g)$$

is well-defined modulo discrete numbers $k\mathbb{Z}$. These discrete ambiguities vanish in the derivation of the classical equations of motion, so that the classical theory is well defined.

To get a well-defined theory on the quantum level as well, the amplitudes $\mathrm{e}^{2\pi \mathrm{i} S(g)}$ in Feynman's path integral have to be well-defined. They are, if the quantization condition $k \in \mathbb{Z}$ is satisfied. Here k is called the level of the model.

The quantization of a theory with the action $S(g)$ gives a theory with conformal invariance. It also possesses an invariance under both right and left translation.

Conserved currents. By Noether's theorem, the invariance under right and left translation gives rise to two conserved currents, they are

$$J(g) = -(1+\star)\, g^*\theta \quad \text{and} \quad \overline{J}(g) = (1-\star)\, g^*\overline{\theta}.$$

Witten observed, that the Wess-Zumino-Witten-model has another symmetry, which he called parity symmetry. That is, a flip of the orientation of Σ together with a transformation $g \longmapsto g^{-1}$ in the target space. Under this symmetry, the two conserved currents are exchanged, for that reason they are sometimes called equivalent.

To make contact with other developments in conformal field theory, the following point of view is useful. We do not choose an orientation on Σ, and consider instead the double covering $\hat{\Sigma}$ with its canonical orientation and involution σ. Consider now maps $\hat{g} : \hat{\Sigma} \longrightarrow G$ which satisfy $\hat{g} = \hat{g}^{-1} \circ \sigma$. Let $C^\infty(\hat{\Sigma}, G)^\sigma$ be the space of such maps.

An orientation on Σ gives a natural identification

$$C^\infty(\hat{\Sigma}, G)^\sigma = C^\infty(\Sigma, G),$$

which identifies also the Wess-Zumino-Witten-model on Σ with a Wess-Zumino-Witten-model restricted to maps $C^\infty(\hat{\Sigma}, G)^\sigma$ on $\hat{\Sigma}$. For the later one, we have $J(\hat{g}) = \overline{J}(\hat{g})$, so there is only one conserved chiral current left. This procedure establishes the correspondence between a full conformal field theory on Σ an a chiral conformal field theory on $\hat{\Sigma}$.

Gerbes. Witten's approach is limited to simple, connected and simply connected Lie groups, in particular $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$ is needed for the existence of the extension $\widetilde{g}$. Gerbes turn out to be useful to overcome this restriction.

There is again an analogous situation in quantum mechanics of charged point particles in the background of a $\mathrm{U}(1)$ gauge field of field strength F. If $\pi_1(M) = 0$, the action of a particle $\phi : S^1 \longrightarrow M$ in such a field is given by

$$S(\phi) = e \int_{D^2} \widetilde{\phi}^* F,$$

where e is the charge of the particle and $\widetilde{\phi}$ is an extension of ϕ into the interior D^2 of S^1. The extension exists because $\pi_0(M) = \pi_1(M) = 0$, and the action is well-defined up to integer multiples of e, if F is closed and defines an integral class. The well-definedness of the path integral $\mathrm{e}^{2\pi \mathrm{i} S(\phi)}$ leads to Diracs quantization condition on the electric charge.

If one can choose a global gauge potential with $\mathrm{d}A = eF$, then the action becomes

$$S(\phi) = \int_{S^1} \phi^* A.$$

If there are obstructions to this approach, one can use a hermitian line bundle with connection to describe the background field. Here the field strength eF is the curvature of the bundle. Then the amplitudes

$$\mathrm{e}^{2\pi \mathrm{i} S(\phi)} = \mathrm{hol}(\phi)$$

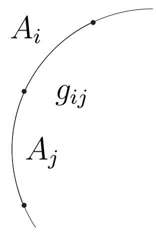
are given by the holonomy of the bundle around ϕ, and one can reproduce both forms of the action.

In this example, the curvature eF does not in general determine the bundle and the connection completely, hence there are more degrees of freedom (→ Aharonov-Bohm-effect). The same is going to happen when generalizing the Wess-Zumino-Witten-model.

In detail, holonomy of a line bundle with connection with local data $(g_{\alpha\beta}, A_\beta)$ with respect to a sufficiently fine open cover of M is given by

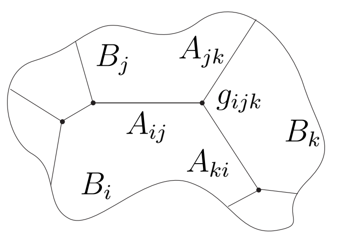
$$\mathrm{hol}_{g,A}(\phi) = \prod_{i \in I} \exp\left(2\pi \mathrm{i} \int_{s_i} A_{\alpha(i)}\right) \cdot \prod_{p \in \partial s} g_{\alpha(i)\alpha(p)}^{\varepsilon(i,p)}(p),$$

where $\{s_i\}_{i \in I}$ is a triangulation of the world line S^1, which is subordinate to the cover, and $\alpha(i)$, $\alpha(p)$ are subordinating indices. This expression coincides via $\mathrm{d}A_\alpha = eF$ with the exponential of the action term.



We first consider local data $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$ with $\mathrm{d}B_\alpha = kH$ and write down an analogous term

$$\mathrm{hol}_{g,A,B}(\phi) = \prod_{i \in I} \exp\left(2\pi \mathrm{i} \int_{\Delta_i} B_{\alpha(i)}\right) \cdot \prod_{e \in \partial \Delta_i} \exp\left(2\pi \mathrm{i} \int_e A_{\alpha(i)\alpha(e)}\right) \cdot \prod_{p \in \partial e} g_{\alpha(i)\alpha(e)\alpha(p)}^{\varepsilon(i,e,p)}(p).$$



This term coincides with the exponential of the Wess-Zumino-term. It turns out to be the holonomy of a gerbe with connective structure, where the threeform kH is the curvature of that gerbe. In general, holonomy of gerbes is defined irrespectively of the values of $\pi_i(M)$, so it is a generalization of the Wess-Zumino-Witten-model to arbitrary target spaces.

In general, a gerbe with connective structure is not uniquely determined by its curvature kH, and the additional degrees of freedom are sometimes called a B-field. The set of non-equivalent gerbes with connective structure of curvature kH is a torsor over $\mathrm{H}^2(M, \mathrm{U}(1))$. This cohomology group can be computed by the universal coefficient theorem

$$\mathrm{H}^2(G, \mathrm{U}(1)) = \mathrm{Hom}(\mathrm{H}_2(G), \mathrm{U}(1)) \oplus \mathrm{Ext}(\mathrm{H}_1(G), \mathrm{U}(1)).$$

Some examples:

- If $M = G$ is a simple, connected, simply connected Lie group like in Witten's approach, every biinvariant, closed threeform with integral class, i.e. every possible curvature of a relevant gerbe, is an integer multiple kH of the threeform H. Furthermore, we have $\mathrm{H}^2(G, \mathrm{U}(1)) = 0$. Hence all gerbes with connective structure, and therefore all Wess-Zumino-Witten-models, are classified by the level k.
- Let again G be of the type considered above, and let Z be its center. Take the Quotient G/Z, which is a non-simply-connected group with $\pi_1(G/Z) = Z$.

  ▼ Take $G = \mathrm{SU}(2)$ with $Z = \mathbb{Z}_2$. On the quotient $\mathrm{SO}(3)$, the closed threeforms with integral class are given by kH for an even level k. As well we have

  $$\mathrm{H}^2(\mathrm{SO}(3), \mathrm{U}(1)) = \mathrm{Ext}(\mathbb{Z}_2, \mathrm{U}(1)) = 0,$$

  so there is for each even level exactly one gerbe.

  ▼ Take $G = \mathrm{Spin}(4n)$ with $Z = \mathbb{Z}_2 \times \mathbb{Z}_2$. The quotient is $\mathrm{SO}(4n)/\mathbb{Z}_2$, and this group has $\mathrm{H}_1(\mathrm{SO}(4n)/\mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. For a n-connected space X and an action of a discrete group Γ we have the exact sequence

  $$0 \longrightarrow \pi_{n+1}(X) \longrightarrow \mathrm{H}_{n+1}(X/\Gamma) \longrightarrow \mathrm{H}_{n+1}(\Gamma) \longrightarrow 0$$

  which relates fundamental groups, homology groups and group homology groups. Applied to $X = G$, $\Gamma = Z$ and $n = 1$ we get

  $$\mathrm{H}_2(\mathrm{SO}(4n)/\mathbb{Z}_2) = \mathrm{H}_2(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2.$$

  Hence we know $\mathrm{H}^2(\mathrm{SO}(4n)/\mathbb{Z}_2, \mathrm{U}(1)) = \mathrm{Hom}(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2$, so this is an example for a Liegroup, which has for a given curvature two possible, non-equivalent gerbes. This matches perfectly results from algebraic computations.

  The appearance of group cohomology when discussing gerbes on discrete quotients is not surprising, because it is relevant for equivariant structures of gerbes on the coverung group G.
- If $M = \mathbb{T}^2$ is the 2-torus, all gerbes are flat, i.e. of curvature zero. Additionaly we have

  $$\mathrm{H}^2(\mathbb{T}, \mathrm{U}(1)) = \mathrm{Hom}(\mathbb{Z}, \mathrm{U}(1)) \oplus \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}, \mathrm{U}(1)) = \mathrm{U}(1),$$

  so all gerbes over the torus are continously parametrized by $\mathrm{U}(1)$. This is the Kalb-Ramond field of string theory.

Transgression. To get a quantum theory for closed strings, one can also consider a closed string $\phi : S^1 \longrightarrow M$ as a point in the loop space LM. As mentioned above, a quantum theory for point particles is given by a hermitian line bundle with connection, so it is natural to consider such a bundle over the loop space.

Transgression maps gerbes on M to line bundles on LM by pullback with the evaluation map $S^1 \times LM \longrightarrow M$ and integration along the fibre S^1.