Parallel Transport Functors of Principal Bundles and (non-abelian) Bundle Gerbes

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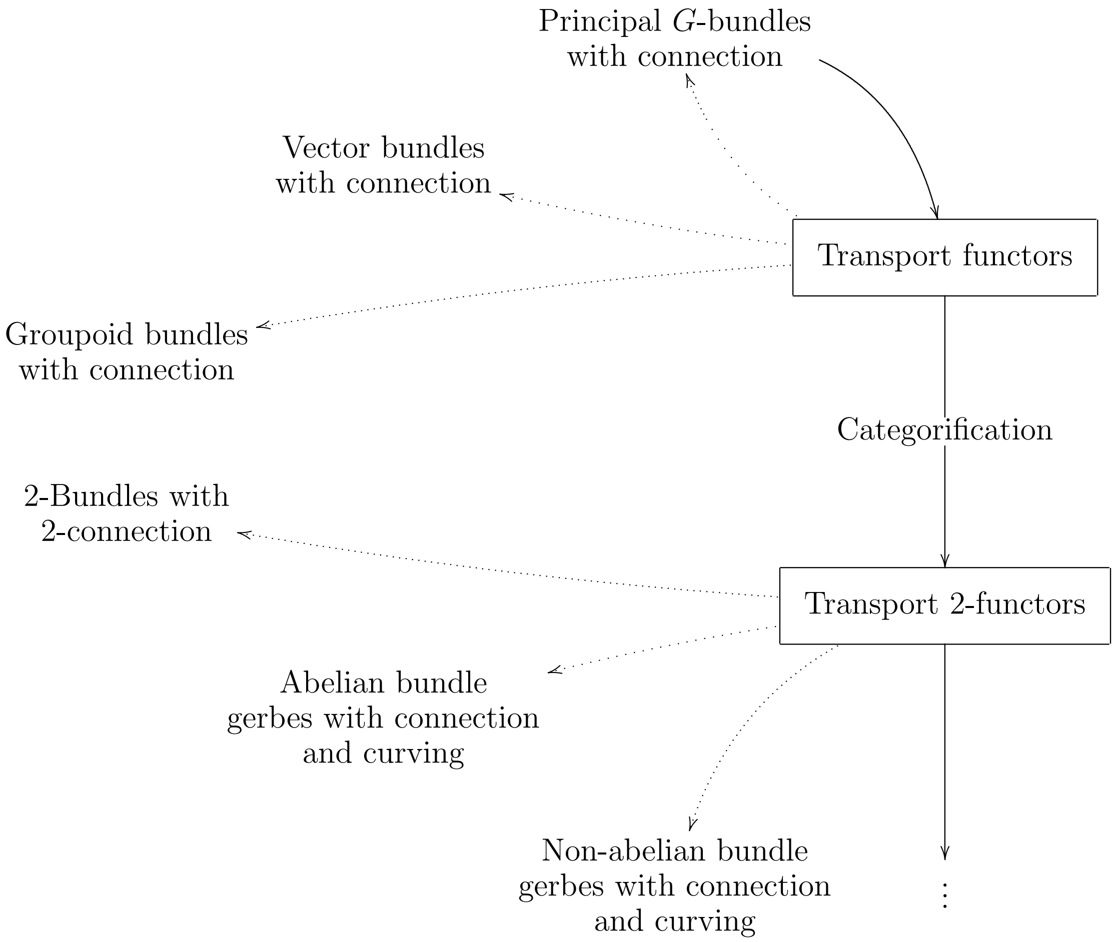
We characterize parallel transport functors defined by connections in principal G-bundles among arbitrary functors by a notion which encodes local triviality and smoothness. In contrast to principal bundles with connection, parallel transport functors admit a natural categorification. This yields a new interpretation of (non-abelian) bundle gerbes with connection and curving as descent data of transport 2-functors.

Plan of the Talk



Ansatz

Let X be a smooth manifold. A principal G-bundle P over X with connection ∇ defines a functor

$$\mathrm{tra}_{P,\nabla} : \mathcal{P}_1(X) \to G\text{-Tor},$$

where:

(1) the *path groupoid* $\mathcal{P}_1(X)$ of X is defined by

$$\mathrm{Obj}(\mathcal{P}_1(X)) := X \quad \text{and} \quad \mathrm{Mor}(\mathcal{P}_1(X)) := PX/\sim\,,$$

PX is the set of smooth maps $\gamma : [0,1] \to X$ with sitting instants, and $\sim$ means *smooth thin homotopy equivalence*.

(2) G-Tor is the category of smooth G-torsors and equivariant smooth diffeomorphisms between those.

Question

We would like to replace principal G-bundles by such functors. For which functors

$$F : \mathcal{P}_1(X) \to G\text{-Tor}$$

exists a principal G-bundle P over X with connection ∇, such that there is a natural isomorphism

$$F \cong \mathrm{tra}_{P,\nabla} \;?$$

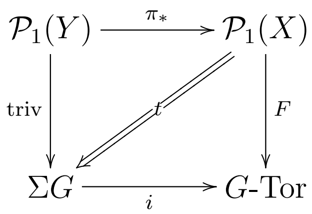
Definition 1

A *local trivialization* of a functor $F : \mathcal{P}_1(X) \to G$-Tor is a triple (π, triv, t) of

1.) a surjective submersion $\pi : Y \to X$

2.) a functor $\mathrm{triv} : \mathcal{P}_1(Y) \to \Sigma G$

3.) a natural isomorphism



Any local trivialization defines a natural isomorphism

$$g := \pi_2^* t \circ \pi_1^* t^{-1} : i \circ \pi_1^*\mathrm{triv} \to i \circ \pi_2^*\mathrm{triv}$$

which satisfies the cocycle condition

$$\pi_{23}^* g \circ \pi_{12}^* g = \pi_{13}^* g.$$

The pair (triv, g) is called the *descent data* of the functor F.

Definition 2

A local trivialization (π, triv, t) of a functor $F : \mathcal{P}_1(X) \to G\text{-Tor}$ is called *smooth*, if the associated descent data (triv, g) is smooth:

1.) The functor $\mathrm{triv} : \mathcal{P}_1(Y) \to \Sigma G$ is smooth.
2.) The natural isomorphism $g : Y^{[2]} \to \mathrm{Mor}(G\text{-Tor})$ factors through a smooth map $\tilde{g} : Y^{[2]} \to G$, i.e. $g := i \circ \tilde{g}$.

Generalization

The notion of a smooth local trivialization makes sense in more general situations, namely when

a) $F : \mathcal{P}_1(M) \to T$ is a functor into an arbitrary category T,
b) Gr is a Lie groupoid, and
c) $i : \mathrm{Gr} \to T$ is a functor.

Definition 3

A functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to T$$

is called *transport functor with* Gr*-structure*, if it admits a smooth local Gr-trivialization

$$\begin{array}{ccc} \mathcal{P}_1(Y) & \xrightarrow{\pi_*} & \mathcal{P}_1(X) \\ {\scriptstyle \mathrm{triv}}\big\downarrow & \overset{t}{\swarrow} & \big\downarrow{\scriptstyle \mathrm{tra}} \\ \mathrm{Gr} & \xrightarrow[i]{} & T. \end{array}$$

Theorem 1

1.) The functor $\mathrm{tra}_{P,\nabla} : \mathcal{P}_1(X) \to G\text{-Tor}$ obtained from parallel transport in a principal G-bundle P with connection ∇ is a transport functor with ΣG-structure.
2.) The functor

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Smooth principal } G\text{-bundles} \\ \text{over } X \text{ with connection} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Transport functors} \\ \mathrm{tra} : \mathcal{P}_1(X) \to G\text{-Tor} \\ \text{with } \Sigma G\text{-structure} \end{array} \right\} \\ (P, \nabla) & \longmapsto & \mathrm{tra}_{P,\nabla} \end{array}$$

is an equivalence of categories.

Sketch of the proof of Theorem 1

to 1.) Any local trivialization $\phi : \pi^*P \to G \times Y$ of the principal G-bundle P defines a smooth local ΣG -trivialization (π, triv, t) of the functor $\text{tra}_{P,\nabla}$: to see this we prove a bijection

$$\Omega^1(Y, \mathfrak{g}) \to \text{Funct}^\infty(\mathcal{P}_1(Y), \Sigma G),$$

by which we obtain the smooth functor $\text{triv} : \mathcal{P}_1(Y) \to \Sigma G$. Then we show that

$$t : Y \to \text{Mor}(G\text{-Tor}) \quad , \quad t(y) := \phi_y : P_{\pi(y)} \to G$$

defines a natural isomorphism $t : \pi^*\text{tra}_{P,\nabla} \to i \circ \text{triv}$. For the descent data we find $g = i \circ \tilde{g}$, where $\tilde{g} : Y^{[2]} \to G$ is the transition function of the principal G-bundle P with respect to the local trivialization ϕ, hence smooth.

to 2) Proof of the essential surjectivity: if $\text{tra} : \mathcal{P}_1(X) \to G\text{-Tor}$ is a transport functor with ΣG-structure, choose a smooth local trivialization. Its descent data (triv, g) determines a 1-form $A \in \Omega^1(Y, \mathfrak{g})$ and a transition function $\tilde{g} : Y^{[2]} \to G$. The principal G-bundle with connection reconstructed from this local data is a preimage of tra.

Examples 1

i) Hermitian vector bundles with unitary connection are obtained (as a monoidal category) by

$$T := \text{Vect}_h(\mathbb{C}) \quad \text{and} \quad \text{Gr} := \bigsqcup_{n \in \mathbb{N}} \Sigma U(n).$$

ii) Transport functors $\text{tra} : \mathcal{P}_1(X) \to \Sigma G$ restricted to thin homotopy classes of loops at a point $p \in X$ give rise to holonomy maps (Barret '91, Caetano-Picken '94)

$$\mathcal{H} : \pi_1^1(M, p) \to G.$$

Categorification

a) We consider transport 2-functors

$$\text{tra} : \mathcal{P}_2(X) \to T$$

with structure Lie 2-groupoids Gr.

b) Descent data are triples (triv, g, f) consisting of a 2-functor $\text{triv} : \mathcal{P}_2(Y) \to \text{Gr}$ and a pseudonatural isomorphism

$$g : i \circ \pi_1^*\text{triv} \to i \circ \pi_2^*\text{triv}$$

which satisfies the cocycle condition up to a coherent modification

$$f : \pi_{23}^*g \circ \pi_{12}^*g \Longrightarrow \pi_{13}^*g.$$

c) Such descent data (triv, g, f) is called *smooth*, if $\text{triv} : \mathcal{P}_2(Y) \to \text{Gr}$ is a smooth 2-functor, if g factors through a transport 1-functor, and if f factors through a morphism of transport 1-functors.

Theorem 2

There is a canonical equivalence of 2-categories:

$$\left\{ \begin{array}{c} \text{Descent data of} \\ \text{transport 2-functors} \\ \mathrm{tra} : \mathcal{P}_2(X) \to \Sigma\mathrm{Vect}^1_h(\mathbb{C}) \\ \text{with } \Sigma\Sigma U(1)\text{-structure} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Hermitian bundle gerbes} \\ \text{with connection and curving} \\ \text{over } X \text{ (Murray '94)} \end{array} \right\}$$

Sketch of the proof of Theorem 2:

Let $\pi : Y \to M$ be a surjective submersion, and (triv, g, f) descent data. Then,

a) the smooth functor $\mathrm{triv} : \mathcal{P}_2(Y) \to \Sigma\Sigma U(1)$ defines a 2-form $C \in \Omega^2(Y)$,
b) the pseudonatural isomorphism $g : \mathcal{P}_1(Y^{[2]}) \to \mathrm{Vect}^1_h(\mathbb{C})$ defines a hermitian line bundle L over $Y^{[2]}$ with connection ∇ of curvature $\pi_2^*C - \pi_1^*C$, and
c) the modification $f : \pi_{23}^*g \circ \pi_{12}^*g \to \pi_{13}^*g$ defines an associative isomorphism $\mu : \pi_{12}^*L \otimes \pi_{23}^*L \to \pi_{13}^*L$ of line bundles over $Y^{[3]}$.

This gives a hermitian bundle gerbe (π, L, μ) with connection ∇ and curving C.

Examples 2

i) (Non-abelian) H-bundle gerbes with connection and curving (Aschieri-Jurco-Cantini '05) are obtained as descent data of transport 2-functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma\mathrm{BiTor}(H)$$

with $\Sigma\mathrm{AUT}(H)$-structure, where

a) $\mathrm{BiTor}(H)$ is the category of H-bi-torsors.
b) $\mathrm{AUT}(H)$ is the Lie 2-group corresponding to the crossed module

$$H \xrightarrow{\mathrm{ad}} \mathrm{Aut}(H) \xrightarrow{\mathrm{id}} \mathrm{Aut}(H)\ .$$

ii) Non-abelian (fake-flat) differential cocycles (Breen-Messing '01) for a Lie 2-group G_2 are obtained as descent data of transport functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma G_2$$

with ΣG_2-structure.