

# Smooth Functors for higher-dimensional Parallel Transport

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# Overview

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without connections in a fibre bundle
4. Evident categorification: Transport 2-functors
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## Motivation: Higher gauge theory

- ▶ Point-like particles: motion along a path  $\gamma : [0, 1] \rightarrow M$  couples to the **parallel transport**

$$\tau_\gamma : E_{\gamma(0)} \rightarrow E_{\gamma(1)}$$

of a connection  $\nabla$  in a fibre bundle  $E$  over  $M$ .

- ▶ String theory: the path  $\gamma$  is replaced by a **surface**  $\phi : \Sigma \rightarrow M$ .
- ▶ Questions:
  1. What is the geometrical structure that replaces the fibre bundle  $E$  and the connection  $\nabla$ ?  
→ "gerbe with connection"
  2. Surfaces can be un-orientable! What are the implications for these gerbes?  
→ "Jandl gerbes" (Schreiber-Schweigert-KW '05)

## Two ways towards higher dimensional parallel transport

- ▶ First way: (Brylinski '93, Murray '95, Breen-Messing '03, Bartels '06, etc.)
  1. Categorify a fibre bundle.
  2. Categorify a connection in a fibre bundle.
  3. Find out what the parallel transport of such a connection is.

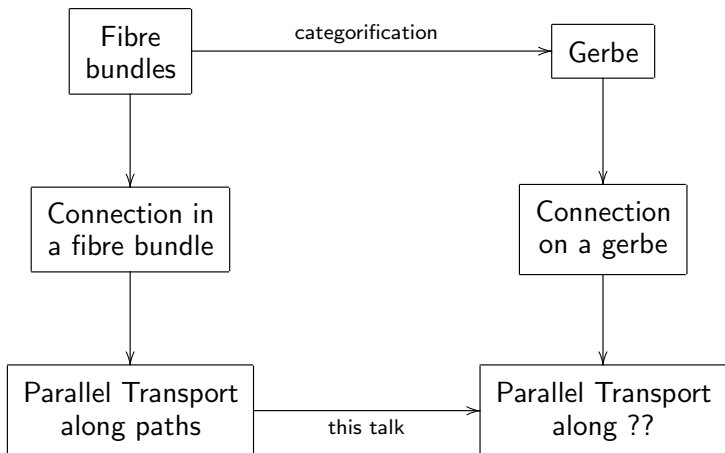
Success: parallel transport along closed surfaces (holonomy) in the "abelian case".

- ▶ Our Alternative (this talk):
  1. Describe the parallel transport of a connection in a fibre bundle **without** using the notion of a connection in a fibre bundle.
  2. Categorify this!

Success: general framework for gerbes with connection and their parallel transport.

# Two ways towards higher dimensional parallel transport

These two ways fit into a "commutative diagram"



# Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

Urs Schreiber, KW "*Parallel Transport and Functors*",  
[arxiv:0705.0452]

Consider a principal  $G$ -bundle  $P$  over  $M$  with connection.

(a) Its parallel transport has the structure of a **functor**

$$F : \mathcal{P}_1(M) \rightarrow G\text{-Tor}$$

between two categories:

1.  $\mathcal{P}_1(M)$  is the **path groupoid** of  $M$ , with
  - ▶ Objects: points of  $M$
  - ▶ Morphisms: thin homotopy classes of smooth paths
2.  $G\text{-Tor}$  is the category of  $G$ -torsors, with
  - ▶ Objects: manifolds with smooth  $G$ -action
  - ▶  $G$ -equivariant smooth maps.

## Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

- (b) Question: how can we characterize parallel transport functors among all functors

$$F : \mathcal{P}_1(M) \rightarrow G\text{-Tor} ?$$

Answer: impose the following two conditions.

1.  $F$  is **locally trivial**
2. Its descent data is **smooth**

We call functors with these properties **transport functors**.

# Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(c) We call a functor

$$F : \mathcal{P}_1(M) \rightarrow G\text{-Tor}$$

locally trivial, if there exist

1. a suitable covering  $\pi : U \rightarrow M$  („surjective submersion“)
2. a functor  $\text{triv} : \mathcal{P}_1(U) \rightarrow G\text{-Tor}$
3. a natural equivalence

$$\begin{array}{ccc} \mathcal{P}_1(U) & \xrightarrow{\pi_*} & \mathcal{P}_1(M) \\ \text{triv} \downarrow & \swarrow \cong & \downarrow F \\ \mathcal{B}G & \xrightarrow{i} & G\text{-Tor} \end{array}$$

with

- ▶  $\mathcal{B}G$  is the groupoid associated to the group  $G$
- ▶  $i : \mathcal{B}G \rightarrow G\text{-Tor}$  is the functor which regards  $G$  as a  $G$ -torsor over itself.



## Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(d) We say that a local trivialization  $(\pi : U \rightarrow M, \text{triv}, t)$  has **smooth descent data**, if

1. the functor

$$\text{triv} : \mathcal{P}_1(U) \rightarrow \mathcal{B}G$$

is **smooth**: internal to the category of **diffeological spaces**.

Key observation: the path groupoid  $\mathcal{P}_1(M)$  is a category internal to diffeological spaces.

2. a certain smoothness condition on  $t$  is satisfied: it comes from a **smooth function**  $g : U \times_M U \rightarrow G$ .

## Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(e) Our results:

Theorem A: There is a canonical equivalence of categories

$$\left\{ \begin{array}{l} \text{Transport functors} \\ F : \mathcal{P}_1(M) \rightarrow G\text{-Tor} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Principal } G\text{-bundles} \\ \text{with connection over } M \end{array} \right\}.$$

Proof: reduce it locally to a statement on *trivial* principal  $G$ -bundles with connection, i.e.  $\mathfrak{g}$ -valued 1-forms:

Theorem B: There is a canonical equivalence of categories

$$\left\{ \begin{array}{l} \text{Smooth functors} \\ \text{triv} : \mathcal{P}_1(U) \rightarrow \mathcal{B}G \end{array} \right\} \cong \Omega^1(U, \mathfrak{g}).$$

Theorem A generalizes further to vector bundles, groupoid bundles...

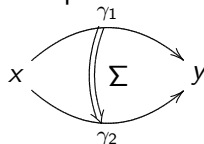
# Evident categorification: Transport 2-functors

Urs Schreiber, KW "Connections in non-abelian Gerbes and their Holonomy", [arxiv:0808.1923]

(a) First step: categorify the **path groupoid**  $\mathcal{P}_1(M)$ .

The path 2-groupoid  $\mathcal{P}_2(M)$  is defined in the following way:

- ▶ Objects: points in  $M$
- ▶ 1-morphisms: thin homotopy classes of smooth paths
- ▶ 2-morphisms: thin homotopy classes of **smooth homotopies** between paths:



These homotopies between paths are the **surfaces** along which we perform parallel transport!

## Evident categorification: Transport 2-functors

(b) Second step: categorify the category  $G\text{-Tor}$ .

For the purposes of this talk, we restrict ourselves to the case of " $S^1$ -gerbes".

Then, we consider 2-functors

$$F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor})$$

with

- ▶  $\mathcal{P}_2(M)$  the path 2-groupoid of  $M$
- ▶  $\mathcal{B}S^1\text{-Tor}$  the 2-category associated to the monoidal category of  $S^1$ -torsors.

## Evident categorification: Transport 2-functors

- (c) Third step: categorify **local triviality** and **smoothness** conditions on the descent data of a 2-functor

$$F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor}).$$

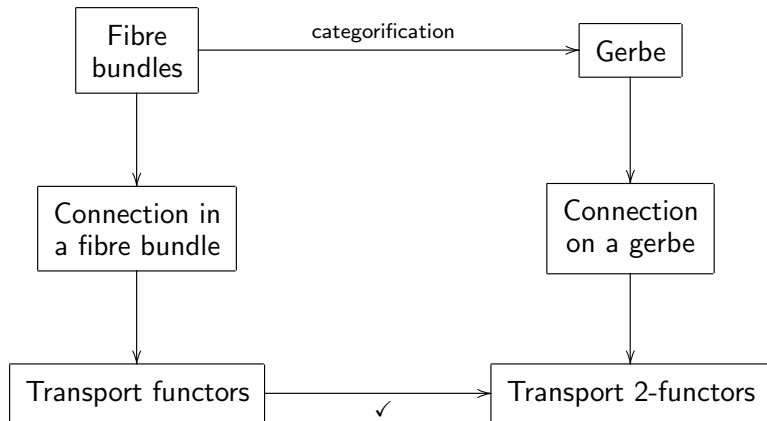
We call these functors **transport 2-functors**.

The conditions imply the existence of

- ▶ a covering  $\pi : U \rightarrow M$
- ▶ a **smooth 2-functor**  $\text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1$
- ▶ a **transport functor**  $g : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor}$
- ▶ ...

## Evident categorification: Transport 2-functors

Question: do **transport 2-functors** make our diagram "commutative" ?



Answer: they do!

## Evident categorification: Transport 2-functors

(d) Our results:

Theorem C: There is a canonical equivalence of 2-categories

$$\left\{ \begin{array}{l} \text{Transport 2-functors} \\ F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor}) \end{array} \right\} \cong \left\{ \begin{array}{l} S^1\text{-bundle gerbes with} \\ \text{connection over } M \end{array} \right\}$$

Proof: translate the descent data  $(\pi, \text{triv}, g, \dots)$  of a transport 2-functor into "geometrical data":

$$\begin{array}{ccc} \begin{array}{l} \text{smooth functor} \\ \text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1 \end{array} & \longmapsto & B \in \Omega^2(U) \\ \begin{array}{l} \text{transport functor} \\ g : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor} \end{array} & \xrightarrow{\text{Thm A}} & \begin{array}{l} \text{Principal} \\ S^1\text{-bundle with} \\ \text{connection over} \\ U \times_M U \end{array} \\ \dots & \longmapsto & \dots \end{array} \left. \vphantom{\begin{array}{l} \text{smooth functor} \\ \text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1 \end{array}} \right\} \begin{array}{l} \text{this is a} \\ \text{bundle} \\ \text{gerbe w.} \\ \text{connec-} \\ \text{tion} \end{array}$$

## Evident categorification: Transport 2-functors

- (e) Further results show that **transport 2-functors** reproduce:
- ▶ non-abelian bundle gerbes
  - ▶ Breen-Messing gerbes
  - ▶ non-abelian differential cohomology



## One Consequence: Holonomy of non-abelian Gerbes

(a) Consider a transport functor  $F : \mathcal{P}_1(M) \rightarrow G\text{-Tor}$ , and an oriented closed line  $S \subset M$ .

- ▶ To compute the **holonomy** of  $F$  around  $S$ , we have to regard  $S$  as a path in  $M$ , i.e. a morphism

$$\gamma : x \rightarrow x$$

in  $\mathcal{P}_1(M)$ , chosen **compatible with the orientation** of  $S$ .

- ▶ The **holonomy** is then  $F(\gamma) \in \text{Mor}(G\text{-Tor})$ .

Remark: unless  $G$  is abelian, it not possible to identify  $F(\gamma)$  with a group element.

- ▶ The holonomy **depends** on the choice of the **base point**  $x \in S$ , but in a "controlled way".

## One Consequence: Holonomy of non-abelian Gerbes

- (b) Consider now a transport 2-functor  $F : \mathcal{P}_2(M) \rightarrow T$ , and an oriented closed surface  $S \subset M$ .
- ▶ To compute the **surface holonomy** of  $F$  around  $\Sigma$ , we have to regard  $S$  as a 2-morphism in  $\mathcal{P}_2(M)$ .
  - ▶ One can always arrange this 2-morphism to be of the form

$$\Sigma : \gamma \Rightarrow \text{id}_x$$

for a **base point**  $x \in S$  and a **closed path**  $\gamma : x \rightarrow x$ .

- ▶ The **surface holonomy** is then  $F(\Sigma) \in 2\text{-Mor}(T)$ .

**Theorem D:** The surface holonomy  $F(\Sigma)$  depends on the choice of a base point  $x$  and of a path  $\gamma$ , but in a "controlled way".

# Conclusions

- ▶ We have formalized the parallel transport of a connection in a fibre bundle, and obtained the concept of a **transport functor**.
- ▶ The categorification of this concept provides an alternative way to understand **gerbes with connection**.
- ▶ It coincides with all known definitions of gerbes with connection, and **prescribes** what exactly the **parallel transport** of a gerbe with connection is.