Smooth Functors for higher-dimensional Parallel Transport

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Overview

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2. Two ways towards higher dimensional parallel transport
3. Parallel transport of a connection in a fibre bundle without connections in a fibre bundle
4. Evident categorification: Transport 2-functors
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Motivation: Higher gauge theory

- Point-like particles: motion along a path $\gamma : [0,1] \to M$ couples to the parallel transport

  $$\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$

  of a connection ∇ in a fibre bundle E over M.

- String theory: the path γ is replaced by a surface $\phi : \Sigma \to M$.

- Questions:

  1. What is the geometrical structure that replaces the fibre bundle E and the connection ∇?
     $\to$ "gerbe with connection"

  2. Surfaces can be un-orientable! What are the implications for these gerbes?
     $\to$ "Jandl gerbes" (Schreiber-Schweigert-KW '05)

Two ways towards higher dimensional parallel transport

- First way: (Brylinski '93, Murray '95, Breen-Messing '03, Bartels '06, etc.)
  1. Categorify a fibre bundle.
  2. Categorify a connection in a fibre bundle.
  3. Find out what the parallel transport of such a connection is.

  Success: parallel transport along closed surfaces (holonomy) in the "abelian case".

- Our Alternative (this talk):
  1. Describe the parallel transport of a connection in a fibre bundle without using the notion of a connection in a fibre bundle.
  2. Categorify this!

  Success: general framework for gerbes with connection and their parallel transport.

Two ways towards higher dimensional parallel transport

These two ways fit into a "commutative diagram"

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Fibre}\\ \text{bundles}\end{array}} \arrow[r, "\text{categorification}"] \arrow[d] & \boxed{\text{Gerbe}} \arrow[d] \\
\boxed{\begin{array}{c}\text{Connection in}\\ \text{a fibre bundle}\end{array}} \arrow[d] & \boxed{\begin{array}{c}\text{Connection}\\ \text{on a gerbe}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\text{Parallel Transport}\\ \text{along paths}\end{array}} \arrow[r, "\text{this talk}"'] & \boxed{\begin{array}{c}\text{Parallel Transport}\\ \text{along ??}\end{array}}
\end{tikzcd}$$

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

Urs Schreiber, KW "*Parallel Transport and Functors*", [arxiv:0705.0452]

Consider a principal G-bundle P over M with connection.

(a) Its parallel transport has the structure of a functor

$$F : \mathcal{P}_1(M) \to G\text{-Tor}$$

between two categories:

1. $\mathcal{P}_1(M)$ is the path groupoid of M, with
   - Objects: points of M
   - Morphisms: thin homotopy classes of smooth paths
2. G-Tor is the category of G-torsors, with
   - Objects: manifolds with smooth G-action
   - G-equivariant smooth maps.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(b) Question: how can we characterize parallel transport functors among all functors

$$F : \mathcal{P}_1(M) \to G\text{-Tor } ?$$

Answer: impose the following two conditions.

1. F is locally trivial
2. Its descent data is smooth

We call functors with these properties transport functors.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(c) We call a functor

$$F : \mathcal{P}_1(M) \to G\text{-Tor}$$

locally trivial, if there exist

1. a suitable covering $\pi : U \to M$ („surjective submersion“)
2. a functor $\mathrm{triv} : \mathcal{P}_1(U) \to G\text{-Tor}$
3. a natural equivalence

$$\begin{tikzcd} \mathcal{P}_1(U) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{triv}"'] & \mathcal{P}_1(M) \arrow[d, "F"] \arrow[dl, Rightarrow, "t"'] \\ \mathcal{B}G \arrow[r, "i"'] & G\text{-Tor} \end{tikzcd}$$

with

- $\mathcal{B}G$ is the groupoid associated to the group G
- $i : \mathcal{B}G \to G\text{-Tor}$ is the functor which regards G as a G-torsor over itself.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(d) We say that a local trivialization $(\pi : U \to M, \mathrm{triv}, t)$ has smooth descent data, if

1. the functor

$$\mathrm{triv} : \mathcal{P}_1(U) \to \mathcal{B}G$$

   is smooth: internal to the category of diffeological spaces.

   Key observation: the path groupoid $\mathcal{P}_1(M)$ is a category internal to diffeological spaces.

2. a certain smoothness condition on t is satisfied: it comes from a smooth function $g : U \times_M U \to G$.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(e) Our results:

Theorem A: There is a canonical equivalence of categories

$$\left\{ \begin{array}{c} \text{Transport functors} \\ F : \mathcal{P}_1(M) \to G\text{-Tor} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{with connection over } M \end{array} \right\}.$$

Proof: reduce it locally to a statement on *trivial* principal G-bundles with connection, i.e. $\mathfrak{g}$-valued 1-forms:

Theorem B: There is a canonical equivalence of categories

$$\left\{ \begin{array}{c} \text{Smooth functors} \\ \mathrm{triv} : \mathcal{P}_1(U) \to \mathcal{B}G \end{array} \right\} \cong \Omega^1(U, \mathfrak{g}).$$

Theorem A generalizes further to vector bundles, groupoid bundles...

Evident categorification: Transport 2-functors

Urs Schreiber, KW "*Connections in non-abelian Gerbes and their Holonomy*", [arxiv:0808.1923]

(a) First step: categorify the path groupoid $\mathcal{P}_1(M)$.

The path 2-groupoid $\mathcal{P}_2(M)$ is defined in the following way:

- Objects: points in M
- 1-morphisms: thin homotopy classes of smooth paths

like for $\mathcal{P}_1(M)$

- 2-morphisms: thin homotopy classes of smooth homotopies between paths:

$$\begin{tikzcd} x \arrow[r, bend left=50, "\gamma_1"{name=U}] \arrow[r, bend right=50, "\gamma_2"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, "\Sigma"] \end{tikzcd}$$

These homotopies between paths are the surfaces along which we perform parallel transport!

Evident categorification: Transport 2-functors

(b) Second step: categorify the category G-Tor.

For the purposes of this talk, we restrict ourselves to the case of "S^1-gerbes".

Then, we consider 2-functors

$$F : \mathcal{P}_2(M) \to \mathcal{B}(S^1\text{-Tor})$$

with

- $\mathcal{P}_2(M)$ the path 2-groupoid of M
- $\mathcal{B}S^1$-Tor the 2-category associated to the monoidal category of S^1-torsors.

Evident categorification: Transport 2-functors

(c) Third step: categorify local triviality and smoothness conditions on the descent data of a 2-functor

$$F : \mathcal{P}_2(M) \to \mathcal{B}(S^1\text{-Tor}).$$

We call these functors transport 2-functors.

The conditions imply the existence of

- a covering $\pi : U \to M$
- a smooth 2-functor $\mathrm{triv} : \mathcal{P}_2(U) \to \mathcal{B}\mathcal{B}S^1$
- a transport functor $g : \mathcal{P}_1(U \times_M U) \to S^1\text{-Tor}$
- ...

Evident categorification: Transport 2-functors

Question: do transport 2-functors make our diagram "commutative" ?

$$
\begin{tikzcd}
\boxed{\text{Fibre bundles}} \arrow[r, "\text{categorification}"] \arrow[d] & \boxed{\text{Gerbe}} \arrow[d] \\
\boxed{\text{Connection in a fibre bundle}} \arrow[d] & \boxed{\text{Connection on a gerbe}} \arrow[d] \\
\boxed{\text{Transport functors}} \arrow[r, "\checkmark"'] & \boxed{\text{Transport 2-functors}}
\end{tikzcd}
$$

Answer: they do!

Evident categorification: Transport 2-functors

(d) Our results:

Theorem C: There is a canonical equivalence of 2-categories

$$\left\{ \begin{array}{c} \text{Transport 2-functors} \\ F : \mathcal{P}_2(M) \to \mathcal{B}(S^1\text{-Tor}) \end{array} \right\} \cong \left\{ \begin{array}{c} S^1\text{-bundle gerbes with} \\ \text{connection over } M \end{array} \right\}$$

Proof: translate the descent data $(\pi, \mathrm{triv}, g, ...)$ of a transport 2-functor into "geometrical data":

$$\left.\begin{array}{ccc} \begin{array}{c}\text{smooth functor} \\ \mathrm{triv} : \mathcal{P}_2(U) \to \mathcal{B}\mathcal{B}S^1\end{array} & \longmapsto & B \in \Omega^2(U) \\ & & \\ \begin{array}{c}\text{transport functor} \\ g : \mathcal{P}_1(U \times_M U) \to S^1\text{-Tor}\end{array} & \overset{\text{Thm A}}{\longmapsto} & \begin{array}{c}\text{Principal} \\ S^1\text{-bundle with} \\ \text{connection over} \\ U \times_M U\end{array} \\ & & \\ ... & \longmapsto & ... \end{array}\right\} \begin{array}{l}\text{this } \textit{is} \text{ a} \\ \text{bundle} \\ \text{gerbe w.} \\ \text{connec-} \\ \text{tion}\end{array}$$

Evident categorification: Transport 2-functors

(e) Further results show that transport 2-functors reproduce:

- non-abelian bundle gerbes
- Breen-Messing gerbes
- non-abelian differential cohomology

One Consequence: Holonomy of non-abelian Gerbes

(a) Consider a transport functor $F : \mathcal{P}_1(M) \to G\text{-Tor}$, and an oriented closed line $S \subset M$.

- To compute the holonomy of F around S, we have to regard S as a path in M, i.e. a morphism

$$\gamma : x \to x$$

in $\mathcal{P}_1(M)$, chosen compatible with the orientation of S.

- The holonomy is then $F(\gamma) \in \mathrm{Mor}(G\text{-Tor})$.

  Remark: unless G is abelian, it not possible to identify $F(\gamma)$ with a group element.

- The holonomy depends on the choice of the base point $x \in S$, but in a "controlled way".

One Consequence: Holonomy of non-abelian Gerbes

(b) Consider now a transport 2-functor $F : \mathcal{P}_2(M) \to T$, and an oriented closed surface $S \subset M$.

- To compute the surface holonomy of F around Σ, we have to regard S as a 2-morphism in $\mathcal{P}_2(M)$.

- One can always arrange this 2-morphism to be of the form

$$\Sigma : \gamma \Rightarrow \mathrm{id}_x$$

  for a base point $x \in S$ and a closed path $\gamma : x \to x$.

- The surface holonomy is then $F(\Sigma) \in 2\text{-}\mathrm{Mor}(T)$.

Theorem D: The surface holonomy $F(\Sigma)$ depends on the choice of a base point x and of a path γ, but in a "controlled way".

Conclusions

- We have formalized the parallel transport of a connection in a fibre bundle, and obtained the concept of a transport functor.
- The categorification of this concept provides an alternative way to understand gerbes with connection.
- It coincides with all known definitions of gerbes with connection, and prescribes what exactly the parallel transport of a gerbe with connection is.