

Smooth Functors for higher-dimensional Parallel Transport

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Overview

1. Motivation: Higher gauge theory
2. Two ways towards higher dimensional parallel transport
3. Parallel transport of a connection in a fibre bundle
without connections in a fibre bundle
4. Evident categorification: Transport 2-functors
5. One consequence: Holonomy of non-abelian gerbes

Motivation: Higher gauge theory

- ▶ Point-like particles: motion along a path $\gamma : [0, 1] \rightarrow M$ couples to the **parallel transport**

$$\tau_\gamma : E_{\gamma(0)} \rightarrow E_{\gamma(1)}$$

of a connection ∇ in a fibre bundle E over M .

- ▶ String theory: the path γ is replaced by a **surface** $\phi : \Sigma \rightarrow M$.
- ▶ Questions:
 1. What is the geometrical structure that replaces the fibre bundle E and the connection ∇ ?
→ "gerbe with connection"
 2. Surfaces can be un-orientable! What are the implications for these gerbes?
→ "Jandl gerbes" (Schreiber-Schweigert-KW '05)

Two ways towards higher dimensional parallel transport

- ▶ First way: (Brylinski '93, Murray '95, Breen-Messing '03, Bartels '06, etc.)
 1. Categorify a fibre bundle.
 2. Categorify a connection in a fibre bundle.
 3. Find out what the parallel transport of such a connection is.

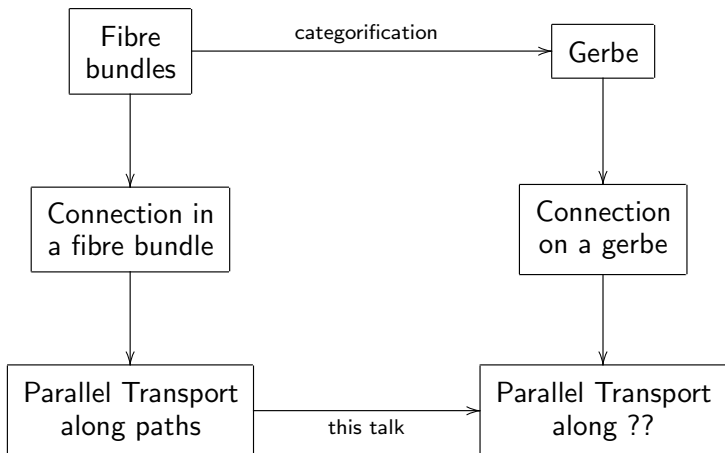
Success: parallel transport along closed surfaces (holonomy) in the "abelian case".

- ▶ Our Alternative (this talk):
 1. Describe the parallel transport of a connection in a fibre bundle **without** using the notion of a connection in a fibre bundle.
 2. Categorify this!

Success: general framework for gerbes with connection and their parallel transport.

Two ways towards higher dimensional parallel transport

These two ways fit into a "commutative diagram"



Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

Urs Schreiber, KW "*Parallel Transport and Functors*",
[arxiv:0705.0452]

Consider a principal G -bundle P over M with connection.

(a) Its parallel transport has the structure of a **functor**

$$F : \mathcal{P}_1(M) \rightarrow G\text{-Tor}$$

between two categories:

1. $\mathcal{P}_1(M)$ is the **path groupoid** of M , with
 - ▶ Objects: points of M
 - ▶ Morphisms: thin homotopy classes of smooth paths
2. $G\text{-Tor}$ is the category of G -torsors, with
 - ▶ Objects: manifolds with smooth G -action
 - ▶ G -equivariant smooth maps.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

- (b) Question: how can we characterize parallel transport functors among all functors

$$F : \mathcal{P}_1(M) \rightarrow G\text{-Tor} ?$$

Answer: impose the following two conditions.

1. F is **locally trivial**
2. Its descent data is **smooth**

We call functors with these properties **transport functors**.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(c) We call a functor

$$F : \mathcal{P}_1(M) \rightarrow G\text{-Tor}$$

locally trivial, if there exist

1. a suitable covering $\pi : U \rightarrow M$ („surjective submersion“)
2. a functor $\text{triv} : \mathcal{P}_1(U) \rightarrow G\text{-Tor}$
3. a natural equivalence

$$\begin{array}{ccc} \mathcal{P}_1(U) & \xrightarrow{\pi_*} & \mathcal{P}_1(M) \\ \text{triv} \downarrow & \swarrow \cong & \downarrow F \\ \mathcal{B}G & \xrightarrow{i} & G\text{-Tor} \end{array}$$

with

- ▶ $\mathcal{B}G$ is the groupoid associated to the group G
- ▶ $i : \mathcal{B}G \rightarrow G\text{-Tor}$ is the functor which regards G as a G -torsor over itself.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(d) We say that a local trivialization $(\pi : U \rightarrow M, \text{triv}, t)$ has **smooth descent data**, if

1. the functor

$$\text{triv} : \mathcal{P}_1(U) \rightarrow \mathcal{B}G$$

is **smooth**: internal to the category of **diffeological spaces**.

Key observation: the path groupoid $\mathcal{P}_1(M)$ is a category internal to diffeological spaces.

2. a certain smoothness condition on t is satisfied: it comes from a **smooth function** $g : U \times_M U \rightarrow G$.

Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(e) Our results:

Theorem A: There is a canonical equivalence of categories

$$\left\{ \begin{array}{l} \text{Transport functors} \\ F : \mathcal{P}_1(M) \rightarrow G\text{-Tor} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Principal } G\text{-bundles} \\ \text{with connection over } M \end{array} \right\}.$$

Proof: reduce it locally to a statement on *trivial* principal G -bundles with connection, i.e. \mathfrak{g} -valued 1-forms:

Theorem B: There is a canonical equivalence of categories

$$\left\{ \begin{array}{l} \text{Smooth functors} \\ \text{triv} : \mathcal{P}_1(U) \rightarrow \mathcal{B}G \end{array} \right\} \cong \Omega^1(U, \mathfrak{g}).$$

Theorem A generalizes further to vector bundles, groupoid bundles...

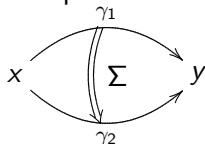
Evident categorification: Transport 2-functors

Urs Schreiber, KW "Connections in non-abelian Gerbes and their Holonomy", [arxiv:0808.1923]

(a) First step: categorify the **path groupoid** $\mathcal{P}_1(M)$.

The path 2-groupoid $\mathcal{P}_2(M)$ is defined in the following way:

- ▶ Objects: points in M
- ▶ 1-morphisms: thin homotopy classes of smooth paths
- ▶ 2-morphisms: thin homotopy classes of **smooth homotopies** between paths:



These homotopies between paths are the **surfaces** along which we perform parallel transport!

Evident categorification: Transport 2-functors

(b) Second step: categorify the category $G\text{-Tor}$.

For the purposes of this talk, we restrict ourselves to the case of " S^1 -gerbes".

Then, we consider 2-functors

$$F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor})$$

with

- ▶ $\mathcal{P}_2(M)$ the path 2-groupoid of M
- ▶ $\mathcal{B}S^1\text{-Tor}$ the 2-category associated to the monoidal category of S^1 -torsors.

Evident categorification: Transport 2-functors

- (c) Third step: categorify **local triviality** and **smoothness** conditions on the descent data of a 2-functor

$$F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor}).$$

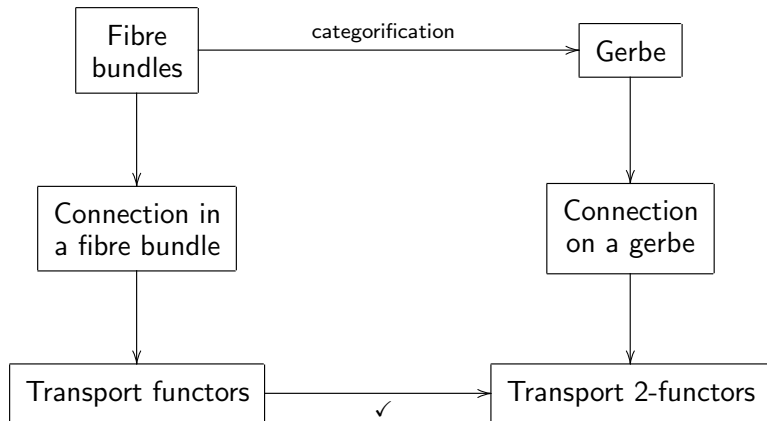
We call these functors **transport 2-functors**.

The conditions imply the existence of

- ▶ a covering $\pi : U \rightarrow M$
- ▶ a **smooth 2-functor** $\text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1$
- ▶ a **transport functor** $g : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor}$
- ▶ ...

Evident categorification: Transport 2-functors

Question: do **transport 2-functors** make our diagram "commutative" ?



Answer: they do!

Evident categorification: Transport 2-functors

(d) Our results:

Theorem C: There is a canonical equivalence of 2-categories

$$\left\{ \begin{array}{l} \text{Transport 2-functors} \\ F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor}) \end{array} \right\} \cong \left\{ \begin{array}{l} S^1\text{-bundle gerbes with} \\ \text{connection over } M \end{array} \right\}$$

Proof: translate the descent data $(\pi, \text{triv}, g, \dots)$ of a transport 2-functor into "geometrical data":

$$\begin{array}{ccc} \begin{array}{l} \text{smooth functor} \\ \text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1 \end{array} & \longmapsto & B \in \Omega^2(U) \\ \begin{array}{l} \text{transport functor} \\ g : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor} \end{array} & \xrightarrow{\text{Thm A}} & \begin{array}{l} \text{Principal} \\ S^1\text{-bundle with} \\ \text{connection over} \\ U \times_M U \end{array} \\ \dots & \longmapsto & \dots \end{array} \left. \vphantom{\begin{array}{l} \text{smooth functor} \\ \text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1 \\ \text{transport functor} \\ g : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor} \\ \dots \end{array}} \right\} \begin{array}{l} \text{this is a} \\ \text{bundle} \\ \text{gerbe w.} \\ \text{connec-} \\ \text{tion} \end{array}$$

Evident categorification: Transport 2-functors

- (e) Further results show that **transport 2-functors** reproduce:
- ▶ non-abelian bundle gerbes
 - ▶ Breen-Messing gerbes
 - ▶ non-abelian differential cohomology

One Consequence: Holonomy of non-abelian Gerbes

(a) Consider a transport functor $F : \mathcal{P}_1(M) \rightarrow G\text{-Tor}$, and an oriented closed line $S \subset M$.

- ▶ To compute the **holonomy** of F around S , we have to regard S as a path in M , i.e. a morphism

$$\gamma : x \rightarrow x$$

in $\mathcal{P}_1(M)$, chosen **compatible with the orientation** of S .

- ▶ The **holonomy** is then $F(\gamma) \in \text{Mor}(G\text{-Tor})$.

Remark: unless G is abelian, it not possible to identify $F(\gamma)$ with a group element.

- ▶ The holonomy **depends** on the choice of the **base point** $x \in S$, but in a "controlled way".

One Consequence: Holonomy of non-abelian Gerbes

- (b) Consider now a transport 2-functor $F : \mathcal{P}_2(M) \rightarrow T$, and an oriented closed surface $S \subset M$.
- ▶ To compute the **surface holonomy** of F around Σ , we have to regard S as a 2-morphism in $\mathcal{P}_2(M)$.
 - ▶ One can always arrange this 2-morphism to be of the form

$$\Sigma : \gamma \Rightarrow \text{id}_x$$

for a **base point** $x \in S$ and a **closed path** $\gamma : x \rightarrow x$.

- ▶ The **surface holonomy** is then $F(\Sigma) \in 2\text{-Mor}(T)$.

Theorem D: The surface holonomy $F(\Sigma)$ depends on the choice of a base point x and of a path γ , but in a "controlled way".

Conclusions

- ▶ We have formalized the parallel transport of a connection in a fibre bundle, and obtained the concept of a **transport functor**.
- ▶ The categorification of this concept provides an alternative way to understand **gerbes with connection**.
- ▶ It coincides with all known definitions of gerbes with connection, and **prescribes** what exactly the **parallel transport** of a gerbe with connection is.