Transgressive central extensions of loop groups

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Let G be a compact connected Lie group, e.g. $G = \mathrm{SU}(2)$.

The **loop group** is the set of smooth loops in G,

$$LG := C^\infty(S^1, G).$$

The group structure is point-wise multiplication.

It is a Fréchet Lie group, with Lie algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.

Unfortunately, LG has no interesting unitary representations.

However, it has *projective-unitary* representations, i.e. it has central extensions

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

and representations $\rho : \mathcal{L} \longrightarrow U(\mathcal{H})$.

Some central extensions

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

have an interesting subclass of representations: **positive-energy representations**.

This class of representations is accessible by “classical” methods: weights, Weyl groups, Borel-Bott-Weil theory,...

Book “Loop groups” by A. Pressley and G. Segal (1986).

Under Connes fusion, positive energy representations form a modular tensor category. This tensor category has nice algebraical descriptions (via VOAs, quantum groups at roots of unity, conformal nets...).

Goal of this talk:

Describe an approach to the representation theory of loop groups via **higher-categorical**, **finite-dimensional** geometry.

Let M be a smooth manifold.

Some examples of **higher-categorical geometry** over M:
gerbes, 2-vector bundles, B-fields, string geometry,...

General slogan (J.-L. Brylinski, 1993):

$$\left\{\begin{matrix}\text{Higher-categorical}\\ \text{geometry over } M\end{matrix}\right\} \xrightarrow{\text{Transgression}} \left\{\begin{matrix}\text{Ordinary geometry}\\ \text{over the loop space}\\ LM = C^\infty(S^1, M)\end{matrix}\right\}$$

$$\begin{matrix}\Cup & & \Cup\\ \text{Gerbe} & \longmapsto & \mathrm{U}(1)\text{-principal bundle}\end{matrix}$$

General phenomenon:

- transgression is not surjective.
- transgression is not injective.

Some details — the **definition of a gerbe**.

Recall: a principal G-bundle P over M can be described by

- open sets $U_\alpha \subseteq M$ that cover M,
- transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$, and
- a cocycle condition: $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$ over $U_\alpha \cap U_\beta \cap U_\gamma$.

A gerbe over M can be described by

- open sets $U_\alpha \subseteq M$ that cover M,
- $\mathrm{U}(1)$-principal bundles $P_{\alpha\beta}$ over $U_\alpha \cap U_\beta$,
- bundle isomorphisms

  $$\mu_{\alpha\beta\gamma} : P_{\alpha\beta} \otimes P_{\beta\gamma} \longrightarrow P_{\alpha\gamma}$$

  over $U_\alpha \cap U_\beta \cap U_\gamma$, and
- a cocycle condition for $\mu_{\alpha\beta\gamma}$ over $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$.

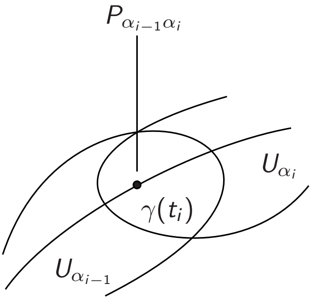
Higher-categorical structure: gerbes form a bicategory.

Some more details — **transgression of a gerbe**.

We define a $\mathrm{U}(1)$-principal bundle $\mathcal{L}$ over LM:

- For a loop $\gamma : S^1 \longrightarrow M$, choose $0 = t_0 \leq ... \leq t_n = 1$ and indices $\alpha_1, ..., \alpha_n$ such that

$$\gamma([t_{i-1}, t_i]) \subseteq U_{\alpha_i}$$



- Define the fibre of $\mathcal{L}$ over γ by

$$\mathcal{L}_\gamma := P_{\alpha_1\alpha_2|_{\gamma(t_1)}} \otimes ... \otimes P_{\alpha_{n-1}\alpha_n|_{\gamma(t_{n-1})}} \otimes P_{\alpha_n\alpha_1|_{\gamma(t_n)}}$$

Isomorphisms $\mu_{\alpha\beta\gamma} \rightsquigarrow$ independence of n and of indices α_i
Connection on $P_{\alpha\beta} \rightsquigarrow$ independence of $t_i \in \gamma^{-1}(U_{\alpha_i} \cap U_{\alpha_{i+1}})$

How is this related to Lie groups?

We put $M := G$ and consider a gerbe over G that is compatible with the group structure ("**multiplicative**").

Multiplicativity is additional structure: if $\mathcal{G}$ is a gerbe over G, it consists of a gerbe isomorphism

$$\mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G} \longrightarrow m^*\mathcal{G}$$

over $G \times G$, and of a certain gerbe 2-isomorphism over $G \times G \times G$ satisfying a coherence condition over $G \times G \times G \times G$.

An example — the **basic gerbe** over $\mathrm{SU}(n)$.

The construction is due to Gawędzki-Reis (2002), and has been generalized by Meinrenken (2002) to arbitrary compact, connected, simple, simply-connected Lie groups.

We choose a maximal torus with Lie algebra $\mathfrak{t}$, a root system and a closed Weyl alcove $\mathfrak{A} \subseteq \mathfrak{t}^*$.

Recall two properties of a Weyl alcove:

- it is a simplex with vertices $0 = \mu_1, ..., \mu_n$.
- it parameterizes conjugacy classes of G.

  This means that there is a (continuous) map

$$q : G \longrightarrow \mathfrak{A}$$

  such that g and $e^{iq(g)}$ are conjugate for every $g \in G$.

Now we write down all the structure of the **basic gerbe** over $\mathrm{SU}(n)$:

1. For $\alpha = 1, ..., n$, define open sets

$$U_\alpha := q^{-1}(\mathfrak{A} \setminus f_\alpha),$$

   where f_α is the face of $\mathfrak{A}$ opposite to the vertex μ_α.

2. There is a deformation retract

$$r : U_\alpha \cap U_\beta \longrightarrow \mathcal{O}_{\alpha\beta}$$

   onto the coadjoint orbit $\mathcal{O}_{\alpha\beta}$ through $\mu_\beta - \mu_\alpha \in \mathfrak{t}^*$.

   The elements $\mu_\beta - \mu_\alpha$ are weights, so that $\mathcal{O}_{\alpha\beta}$ is quantizable in the sense of symplectic geometry.

   Define $P_{\alpha\beta}$ as the pullback of Kirillov-Kostant-Souriau prequantum bundle along the retract.

3. The isomorphism $\mu_{\alpha\beta\gamma}$ comes from the equality

$$\mu_\gamma - \mu_\alpha = (\mu_\beta - \mu_\alpha) + (\mu_\gamma - \mu_\beta).$$

In the multiplicative context, transgression becomes a map

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{Multiplicative gerbes}\\ \text{over G}\end{array}\right\} & \xrightarrow{\text{Transgression}} & \left\{\begin{array}{c}\text{Central extensions}\\ \text{of } LG\end{array}\right\} \\
\downarrow & & \downarrow \\
\mathrm{H}^4(BG,\mathbb{Z}) & \xrightarrow[\int_{S^1}\mathrm{ev}^*]{} & \mathrm{H}^3(BLG,\mathbb{Z}) \cong \mathrm{H}^2(BLG,\underline{\mathrm{U}(1)})
\end{array}$$

In the case of $G = \mathrm{SU}(n)$, this diagram becomes the following:

$$\begin{array}{ccc}
\text{Basic gerbe} & \longmapsto & \begin{array}{c}\text{Universal central}\\ \text{extension of } L\mathrm{SU}(n)\end{array} \\
\downarrow & & \downarrow \\
1 & \longmapsto & 1 \\
\cap & & \cap \\
\mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Z}
\end{array}$$

A central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

is called **transgressive**, if it is in the image of transgression.

Question: given a Lie group G, which central extensions of LG are transgressive?

In other words, which central extensions of LG (in particular, which projective representations) are accessible via higher-categorical geometry?

Some result about **transgressivity**.

Again $G = \mathrm{SU}(n)$. We have seen that the universal central extension is transgressive: it is the image of the basic gerbe under transgression.

Hence, all central extensions of $L\mathrm{SU}(n)$ are transgressive.

This generalizes to all compact, simple, connected Lie groups G.

J.-L. Brylinski & D. McLaughlin (1993-1996) characterized transgressive central extensions, for *complex* Lie groups, in terms of a "Segal-Witten reciprocity law".

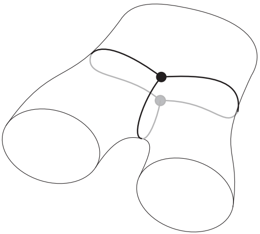
They also proposed a solution for compact Lie groups, but that turned out to be false (noticed around 2000 by Brylinski himself).

Theorem [KW, 2015]

Let G be a connected Lie group. Then, a central extension $\mathcal{L}$ of LG is transgressive if and only if it can be equipped with:

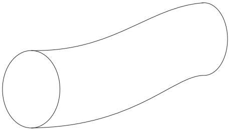
(1) Fusion product — for 3 arcs in G connecting two points, a group homomorphism

$$\mathcal{L}_{\text{left leg's loop}} \otimes \mathcal{L}_{\text{right leg's loop}} \longrightarrow \mathcal{L}_{\text{hip's loop}}$$



(2) Thin homotopy equivariant structure — for a hose in G "without area", a group homomorphism

$$\mathcal{L}_{\text{ingoing loop}} \longrightarrow \mathcal{L}_{\text{outgoing loop}}$$



(+ several conditions)

As a by-product of this characterization, one can deduce two consequences of transgressivity. The first is the following:

Every transgressive central extension is **equivariant under loop rotation**. (This is necessary for imposing positive energy.)

This is proved as follows: let $\tau : S^1 \longrightarrow G$ be a loop and ϕ be an angle. Define $\gamma : [0,1] \longrightarrow LG$ by the formula

$$\gamma(t)(z) := \tau(ze^{it\phi}),$$

so that γ is a path from τ to the rotated loop $rot_\phi(\tau)$. As a map

$$[0,1] \times S^1 \longrightarrow G$$

it has only rank one, i.e. it has "no area". The thin homotopy equivariant structure provides the required lift

$$\mathcal{L}_\tau \longrightarrow \mathcal{L}_{rot_\phi(\tau)}.$$

The second consequence is the following:

Every transgressive central extension is **disjoint commutative** in the following sense.

Suppose loops $\tau_1, \tau_2 : S^1 \longrightarrow G$ have disjoint support, and $\ell_1 \in \mathcal{L}_{\tau_1}$, $\ell_2 \in \mathcal{L}_{\tau_2}$. Then,

$$\ell_1 \cdot \ell_2 = \ell_2 \cdot \ell_1.$$

In particular, for $\rho : \mathcal{L} \longrightarrow U(\mathcal{H})$ a positive-energy representation, the operators $\rho(\ell_1)$ and $\rho(\ell_2)$ commute in $U(\mathcal{H})$.

This is of importance in algebraic quantum field theory formulations of CFT, and was proved for $G = \mathrm{SU}(n)$ by Gabbiani & Fröhlich (1993) via a concrete calculation in the Mickelsson model of the central extension.

Another example: $G = \mathrm{U}(1)$.

Some central extensions of the loop group $L\mathrm{U}(1)$ are transgressive, others are not.

Over $\mathrm{U}(1)$ there is only a single gerbe: the trivial one.

Gerbe isomorphisms between trivial gerbes are just principal $\mathrm{U}(1)$-bundles. Thus, the trivial gerbe becomes multiplicative by specifying a principal $\mathrm{U}(1)$-bundle

$$\begin{array}{c} P \\ \downarrow \\ \mathrm{U}(1) \times \mathrm{U}(1) \end{array}$$

(plus some isomorphism over $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$). There is an interesting choice: the Poincaré bundle.

Under transgression, this yields a non-trivial, transgressive central extension of $L\mathrm{U}(1)$.

On the other hand, one can explicitly write down a smooth 2-cocycle

$$\eta : L\mathrm{U}(1) \times L\mathrm{U}(1) \longrightarrow \mathrm{U}(1)$$

that gives rise to a central extension which is not disjoint-commutative.

Hence it is not transgressive.

This is an example of a central extension that is not accessible via higher-categorical geometry over $\mathrm{U}(1)$.

Summary:

- For every compact connected Lie group G, we have a map

$$\left\{\begin{matrix}\text{Multiplicative gerbes}\\ \text{over G}\end{matrix}\right\} \xrightarrow{\text{Transgression}} \left\{\begin{matrix}\text{Central extensions}\\ \text{of } LG\end{matrix}\right\}$$

- Transgressive central extensions are characterized by a fusion product and a thin homotopy equivariant structure.
- Important central extensions are transgressive, e.g. universal ones.
- This approach explains rotation-equivariance and disjoint commutativity, as derived concepts.

Main message of this talk:

Higher-categorical geometry is useful for understanding loop group extensions and, perhaps in the future, their representation theory.

Why can this be expected?

Freed-Hopkins-Teleman (2003-2010):

$$K_G^{k+h^\vee}(G) \cong \mathrm{Rep}^k(LG)$$

Following a philosophy of Witten (1998), $K_G^{k+h^\vee}(G)$ classifies symmetric D-branes in the level k WZW model over G.

These, in turn, can be described by higher-geometrical structure, Kapustin (2001), Gawędzki-Reis (2002), Carey et al. (2002), Gawędzki (2005).

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