String geometry and spin geometry on loop spaces

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— two approaches to anomaly cancellation in supersymmetric sigma models

1.) Anomalies in supersymmetric sigma models

2.) String geometry

3.) Spin structures on loop spaces

4.) Transgression – from string geometry to spin geometry

The supersymmetric sigma model:

- target space: Riemannian manifold M.
- world sheet: Riemann surface Σ with a spin structure $\mathbb{S}$.
- world sheet embeddings:

$$\phi \in C^\infty(\Sigma, M)$$

- spinors:

$$\psi \in L^2(\Sigma, \mathbb{S} \otimes \phi^* TM)$$

Origin of the anomaly: give sense to the fermionic path integral

$$\mathcal{A}(\phi) = \int_{\psi} \mathrm{D}\psi \, \mathsf{exp}\left(\int_{\Sigma} \langle \psi, \not{D}_{\phi} \psi \rangle \, \mathrm{d}\mathit{vol}_{\Sigma} \right).$$

Well-known solution: $\mathcal{A}(\phi)$ is a well-defined element in a Pfaffian line bundle:

$$\begin{array}{ccc} \mathcal{A}(\phi) & \in & \mathit{Pfaff}\,(\not{D}) \\ \Big\downarrow & & \Big\downarrow \\ \phi & \in & C^{\infty}(\Sigma, M) \end{array}$$

Integrand of the bosonic path integral is not a function, but a section in a complex line bundle – anomaly!

Anomalies of this kind are treated by the Green-Schwarz anomaly mechanism:

1.) Make sure that the line bundle is trivializable.

2.) Provide, for all worldsheets Σ, a trivialization.

By the formula

$$\text{"Section – trivialization} = \text{smooth function"}$$

the integrand of the path integral becomes a smooth function.

Step 1 – make sure that $Pfaff(\not{D})$ is trivializable.

Theorem (Freed '86)

If M is a spin manifold, then

$$c_1(Pfaff(\not{D})) = \int_\Sigma ev^*(\tfrac{1}{2}p_1(M))$$

where

- $ev : C^\infty(\Sigma, M) \times \Sigma \longrightarrow M$ *is the evaluation map*
- $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$ *is the first fractional Pontryagin class*

Sufficient condition: $\frac{1}{2}p_1(M) = 0$.

Spin manifold that satisfy this condition are called string manifolds.

Step 2 of the Green-Schwarz mechanism:

– provide a trivialization of $Pfaff(\not{D})$.

In order to provide such a trivialization consistently for all worldsheets Σ, two interesting geometric theories have been developed:

- Spin geometry on the loop space $LM := C^\infty(\Sigma, M)$
  Witten '86, Killingback '87, Alvarez et al. '87,...
- String geometry on M
  Stolz-Teichner '03, Sati-Schreiber-Stasheff '09,...

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Main idea of string geometry:

$$\frac{1}{2}p_1(M) \in \mathrm{H}^4(M,\mathbb{Z})$$

is the "level" of a Chern-Simons theory over M: we need a notion of a "trivialization" of this Chern-Simons theory that

- exists if and only if M is a string manifold
- it induces a trivialization of $Pfaff(\not{D})$.

Stolz-Teichner proposed to use certain extended field theories, where a trivialization is a 2-dimensional twisted field theory. A precise definition of these notions is in process to be developed.

Here we describe fields theories in terms of the gauge fields: connections on (higher) gerbes.

A short (and informal) reminder on n-gerbes and connections:

0-gerbes $=$ S^1-bundles

- open cover U_α and transition functions $g_{\alpha\beta}$
- connections: local 1-forms A_α
- classified by Chern class in $\mathrm{H}^2(X, \mathbb{Z})$

(1-)gerbes, a.k.a. "B-fields"

- open cover and transition line bundles $+$ higher structure
- connections: local 2-forms $+$ connections on transition bundles
- classified by "Dixmier-Douady class" in $\mathrm{H}^3(X, \mathbb{Z})$

2-gerbes

- open cover and transition gerbes $+$ higher structure
- connections: local 3-forms, connections on transition gerbes,...
- classified by a nameless characteristic class in $\mathrm{H}^4(X, \mathbb{Z})$

Example 1 – Basic gerbe $\mathcal{G}_{bas}$ over a compact, simple, connected simply-connected Lie group G.

(Meinrenken '02, Gawędzki-Reis '02)

- conjugation-invariant open sets U_α corresponding to open subsets of the Weyl alcove, with α the vertices of the alcove.
- $U_\alpha \cap U_\beta$ deformation retracts to the coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ through $\alpha - \beta$. This orbit is integrable: we pull back the prequantum line bundle with its Kostant connection.
- the Dixmier-Douady class is a generator of $\mathrm{H}^3(G, \mathbb{Z}) = \mathbb{Z}$.
- the curvature is the bi-invariant closed 3-form H corresponding to the trilinear form $\langle X, [Y, Z] \rangle$.

In terms of field theories, the basic gerbe $\mathcal{G}_{bas}$ corresponds to the level $k = 1$ Wess-Zumino-Witten model on G.

Example 2 – Chern-Simons 2-gerbe $\mathcal{CS}_M$ over a spin manifold (Carey et al. '05, KW '07)

- Let FM be the frame bundle of M, with its structure group reduced to $\mathrm{Spin}(n)$. Use the projection $FM \longrightarrow M$ as the "open cover".
- The transition gerbe is the pullback of the basic gerbe $\mathcal{G}_{bas}$ along the "difference map" $\delta : FM \times_M FM \longrightarrow \mathrm{Spin}(n)$.
- The local 3-form of the connection is the Chern-Simons 3-form of the Levi-Civita-connection A,

$$\langle A \wedge \mathrm{d}A \rangle + \frac{2}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(FM).$$

- Its characteristic class is $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$.

In terms of field theories, the Chern-Simons 2-gerbe $\mathcal{CS}_M$ corresponds to the Chern-Simons theory over M with level $\frac{1}{2}p_1(M)$.

Two more facts about n-gerbes:

- For every n-gerbe, there is a notion of a trivialization, such that trivializations exist if and only if the characteristic class vanishes.
- Moreover, if the n-gerbe is equipped with a connection, then there is a notion of connections on the trivialization (additional structure for $n > 0$).

Definition

- a string structure on M is a trivialization of the Chern-Simons 2-gerbe $\mathcal{CS}_M$.
- a string connection is a connection on the string structure.
- a geometric string structure is the pair of a string structure and a string connection.

Result 1 – Existence of string connections

Theorem (KW '09)

Every string structure admits a string connection. Moreover, the set of string connections on a fixed string structure is affine.

As a consequence, we obtain the following equivalences:

$$\begin{aligned} \tfrac{1}{2}p_1(M) = 0 \quad &\Longleftrightarrow \quad M \text{ admits a string structure} \\ &\Longleftrightarrow \quad M \text{ admits a geometric string structure.} \end{aligned}$$

Thus, geometric string structures complete Step 1 in the Green-Schwarz mechanism.

Result 2 – Anomaly cancellation

Theorem (Bunke '10)

Every geometric string structure determines a trivialization of the line bundle Pfaff $(\not{D})$.

This result is proved by performing a detailed analysis of the index theory of the Pfaffian line bundle. It is a remarkable line between higher-categorical geometry and classical analysis.

By the theorem, geometric string structures complete Step 2 in the anomaly cancellation mechanism.

In other words, the supersymmetric sigma model requires to fix a geometric string structure on its target space M.

Result 3 – Classification of string structures

- Equivalence classes of string structures are parameterized by

$$\mathrm{H}^3(M, \mathbb{Z}) \cong \left\{ \begin{array}{c} \text{Isomorphism classes} \\ \text{of gerbes over } M \end{array} \right\}$$

- Equivalence classes of geometric string structures are parameterized by the differential cohomology group

$$\hat{\mathrm{H}}^3(M, \mathbb{Z}) \cong \left\{ \begin{array}{c} \text{Isomorphism classes of gerbes} \\ \text{with connection over } M \end{array} \right\}$$

In particular, 2-forms $B \in \Omega^2(M)$ (connections on the trivial gerbe) act on the string connections. Under this action, the trivialization of $\mathit{Pfaff}(\not{D})$ changes by

$$\exp 2\pi \mathrm{i} \int_\Sigma B.$$

In particular, the trivialization depends on the string connection.

Result 4 – The covariant derivative of a string connection

Every geometric string structure on M determines a 3-form $K \in \Omega^3(M)$ with $\mathrm{d}K = \frac{1}{2}\langle F_A \wedge F_A \rangle$.

The Pfaffian line bundle $Pfaff(\not{D})$ comes equipped with the Bismut-Freed connection. The trivialization of $Pfaff(\not{D})$ has covariant derivative

$$\int_\Sigma ev^* K \in \Omega^1(C^\infty(\Sigma, M)).$$

In particular, the trivialization is not parallel.

Höhn-Stolz conjecture: if $\mathrm{Ric}_g > 0$ and $K = 0$, then the Witten genus of M vanishes in $tmf^{-n}(pt)$.

Result 5 – The string 2-group

String structures can also be understood in terms of a (higher) reduction problem in non-abelian gerbes.

There is a central extension

$$B\mathrm{U}(1) \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n)$$

of Lie 2-groups, and one can try to "reduce" the frame bundle FM to a non-abelian gerbe with structure 2-group $\mathrm{String}(n)$.

Theorem (KW-Nikolaus '12)

The Chern-Simons 2-gerbe is the (higher) lifting gerbe of this reduction problem, i.e. there is a 1:1 correspondence between string structures and reductions of FM to $\mathrm{String}(n)$.

An analogous understanding of string connections has not been developed so far.

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We come to the second approach to Step 2 of the Green-Schwarz mechanism

— spin geometry on the loop space $LM = C^\infty(\Sigma, M)$.

Motivation: a string in M is a point in LM, and supersymmetric point-particles are well-understood and treated with spin geometry.

Why still interesting? Many aspects of string geometry are open:

- representation theory of the string 2-group
- the analog of the spinor bundle ("stringor bundle")
- the analog of the Dirac operator and its index (*tmf*-valued?)

Hope: "higher-categorical geometry" can benefit from the well-developed "classical geometry" of the loop space.

Main idea: the class

$$\lambda := \int_{S^1} \mathrm{ev}^* \left(\frac{1}{2} p_1(M) \right) \in \mathrm{H}^3(LM, \mathbb{Z})$$

is the analog of the 3rd Stiefel-Whitney class for the loop space, and can be treated like an obstruction against $\mathrm{Spin}^c(n)$-structures.

The frame bundle of LM is LFM, which is a principal $L\mathrm{Spin}(n)$-bundle over LM.

Theorem (Killingback '87; McLaughlin '92)

λ vanishes if and only if the structure group of LFM lifts to the universal loop group extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{L\mathrm{Spin}(n)} \longrightarrow L\mathrm{Spin}(n) \longrightarrow 1.$$

Definition

A spin structure on LM is a lift of the structure group of LFM from $L\mathrm{Spin}(n)$ to $\widehat{L\mathrm{Spin}(n)}$.

The Levi-Civita connection A on M defines a "looped" connection on LFM. A corresponding lift of this connection is a called spin connection, and the pair of a spin structure and a spin connection is called geometric spin structure.

One can show (Manoharan '02) that every spin structure admits a spin connection. Hence, we have an equivalence

$$\lambda = 0 \quad \Longleftrightarrow \quad LM \text{ admits a geometric spin structure}$$

Two problems:

- $\frac{1}{2}p_1(M) = 0 \Longrightarrow \lambda = 0$ but no equivalence (Pilch-Warner '88)
- it is not clear how geometric spin structures provide trivializations of $Pfaff(\not{D})$.

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Theorem (Murray '95)

For every lifting problem there exists a gerbe ("lifting gerbe") whose trivializations are precisely the possible lifts.

The lifting gerbe $\mathcal{S}_{LM}$ for spin structures over LM ("spin lifting gerbe") is the following:

- its "open cover" is $LFM \longrightarrow LM$.
- its transition bundle is the pullback of

$$\begin{array}{c} L\widehat{\mathrm{Spin}(n)} \\ \big\downarrow \\ L\mathrm{Spin}(n) \end{array}$$

  along $L\delta : LFM \times_{LM} LFM \longrightarrow L\mathrm{Spin}(n)$
- A local connection 2-form can be constructed using the Levi-Civita connection A and a twisted Higgs field (Gomi '03).

Theorem

The spin lifting gerbe $\mathcal{S}_{LM}$ is the transgression of the Chern-Simons 2-gerbe $\mathcal{CS}_M$.

Main ingredients of the proof:

- transgression of gerbes to the loop space (Brylinski '93) takes the basic gerbe $\mathcal{G}_{bas}$ to the universal central extension; this gives coincidence of the transition bundles.
- transgression of the Chern-Simons 3-form is the local connection 2-form of the spin lifting gerbe (Coquereaux-Pilch '98).

There is an induced functor on categories of trivializations:

$$\left\{ \begin{array}{c} \text{Geometric string} \\ \text{structures on } M \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Geometric spin} \\ \text{structures on } LM \end{array} \right\}$$

Thus: string geometry transgresses to spin geometry on LM.

The fact that the vanishing of $\frac{1}{2}p_1(M)$ is not equivalent to the vanishing of λ corresponds to the fact that the functor

$$\left\{ \begin{array}{c} \text{Geometric string} \\ \text{structures on } M \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Geometric spin} \\ \text{structures on } LM \end{array} \right\}$$

is neither injective nor surjective.

The problem can be traced back to the fact that Brylinski's transgression functor

$$\mathcal{G}rb^{\nabla}(M) \longrightarrow \mathcal{B}un^{\nabla}(LM)$$

is neither injective nor surjective.

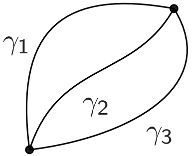
Solution: add additional structure on the loop space side in such a way that the transgression functor becomes an equivalence of categories.

We consider the following additional structures on a line bundle P over LM:

- loop fusion – an associative rule

$$P_{\gamma_1 \cup \gamma_2} \otimes P_{\gamma_2 \cup \gamma_3} \longrightarrow P_{\gamma_1 \cup \gamma_3}$$

relating the fibres of P over the three loops obtained from the figure.



- thin homotopy equivariance – if two loops τ_1 and τ_2 are thin homotopic (homotopic via a rank-one-homotopy), then there are coherent maps $P_{\tau_1} \longrightarrow P_{\tau_2}$ between the fibres of P.
- superficial connections – connections on P whose parallel transport along a thin homotopy gives the thin homotopy equivariant structure.

These structure lead to new categories of line bundles over LM:

- $\mathcal{F}us\mathcal{B}un^{th}(LM)$ – line bundles equipped with a fusion product and a thin homotopy equivariant structure.
- $\mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM)$ – line bundles equipped with a fusion product and a superficial connection.

Theorem (KW '10)

There is a commutative diagram of categories and functors

$$\begin{array}{ccc} \mathcal{G}rb^{\nabla}(M) & \longleftrightarrow & \mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM) \\ \downarrow & & \downarrow \\ \mathcal{G}rb(M) & \longleftrightarrow & \mathcal{F}us\mathcal{B}un^{th}(LM) \end{array}$$

whose horizontal arrows are equivalences, and whose vertical arrows forget the connections (and only keep the induced thin homotopy equivariant structure).

A corresponding modification can be performed with spin structures and spin connections over loop spaces.

Theorem (KW '14)

There is a commutative diagram of categories and functors:

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \textit{Geometric string} \\ \textit{structures over } M \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \textit{Fusion spin structures} \\ \textit{with superficial and} \\ \textit{fusive spin connections} \end{array} \right\} \\
\downarrow & & \downarrow \\
\left\{ \begin{array}{c} \textit{String structures} \\ \textit{over } M \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \textit{Thin fusion spin} \\ \textit{structures} \end{array} \right\}
\end{array}$$

whose horizontal arrows are equivalences.

Conclusions:

- String geometry provides new geometric structures suitable for the anomaly cancellation in supersymmetric sigma models.
- Spin geometry on loop spaces is a similar attempt using classical geometry on the loop space; however, it fails to correctly perform the cancellation mechanism.
- If spin geometry is coupled to loop fusion and thin homotopies, the two geometries become equivalent.

Thank you very much!

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