Lectures on gerbes with connections

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1 Lecture I: Introduction

1.1 Hierarchy of n-gerbes

We work over a (smooth) manifold M, and with S^1 as the "structure group".

n	n-gerbe	classification
-2	$\mathbb{Z}$-valued function	$\mathrm{H}^0(M,\mathbb{Z})$
-1	S^1-valued function	$\mathrm{H}^1(M,\mathbb{Z})$, up to homotopy
0	S^1-bundle	$\mathrm{H}^2(M,\mathbb{Z})$, up to bundle isomorphism
1	S^1-gerbe	$\mathrm{H}^3(M,\mathbb{Z})$, up to gerbe isomorphism
2	S^1-2-gerbe	$\mathrm{H}^4(M,\mathbb{Z})$, up to 2-gerbe isomorphism
n	S^1-n-gerbe	$\mathrm{H}^{n+2}(M,\mathbb{Z})$, up to n-gerbe isomorphism

We start with recalling how to get the structure of an $(n+1)$-gerbe out of an n-gerbe., in the first two case $n=-2$ and $n=-1$.

- An S^1-valued function on M is a cover $\{U_\alpha\}_{\alpha\in A}$ of M by open sets U_α, together with a map $f_\alpha : U_\alpha \cap U_\beta \longrightarrow \mathbb{Z}$ such that $f_{\beta\gamma}\cdot f_{\alpha\beta} = f_{\alpha\beta}$.

Indeed, if $g : M \longrightarrow S^1$ is given, choose the open sets U_α such that $g|_{U_\alpha}$ admits a logarithm $\ln_\alpha(g) : U_\alpha \longrightarrow \mathbb{R}$. On the overlaps, the difference between the branches is $\mathbb{Z}$-valued. Conversely, if $(U_\alpha, g_{\alpha\beta})$ is given, we extend $g_{\alpha\beta}$ to $\mathbb{R}$. Then we have a Čech 1-cocycle with values in the sheaf of smooth $\mathbb{R}$-valued functions, and there exist $k_\alpha : U_\alpha \longrightarrow \mathbb{R}$ with $g_{\alpha\beta} = k_\beta - k_\alpha$. Exponentiating gives $e^{k_\alpha} = e^{k_\beta}$, so that e^k is a well-defined, S^1-valued map on M.

- An S^1-bundle over M is a cover $\{U_\alpha\}_{\alpha\in A}$ of M by open sets U_α, together with a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow S^1$ such that

$$g_{\beta\gamma} \cdot g_{\alpha\beta} = g_{\alpha\gamma}.$$

Remark 1.1.1. In the previous "definition" if an S^1-bundle, we should – taking the table above seriously – actually take maps $g_{\alpha\beta}$ up to homotopy. The fact that we do not do this, comes from a slight misconception in the table. The groups that are actually relevant are the Čech cohomology groups $\check{\mathrm{H}}^{n+1}(M, \underline{S^1})$ with values in the sheaf of smooth, S^1-valued functions. The translation between these and the groups in the table is via the Bockstein of the exponential sequence

$$\check{\mathrm{H}}^{n+1}(M, \underline{S^1}) \longrightarrow H^{n+2}(M, \mathbb{Z}).$$

This homomorphism is an isomorphism only if $n > -1$.

We extrapolate the following definition of the next instance, an S^1-gerbe:

Definition 1.1.2. *An S^1-gerbe over M is a cover $\{U_\alpha\}_{\alpha\in A}$ of M by open sets U_α, together S^1-bundles $P_{\alpha\beta}$ over $U_{\alpha\beta}$, and bundle isomorphisms*

$$\mu_{\alpha\beta\gamma} : P_{\beta\gamma} \otimes P_{\alpha\beta} \longrightarrow P_{\alpha\gamma}$$

over 3-fold intersections $U_\alpha \cap U_\beta \cap U_\gamma$ that are associative in the sense that the diagram

$$\begin{array}{ccc} P_{\gamma\delta} \otimes P_{\beta\gamma} \otimes P_{\alpha\beta} & \xrightarrow{\mu_{\beta\gamma\delta} \otimes \mathrm{id}} & P_{\beta\delta} \otimes P_{\alpha\beta} \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes \mu_{\alpha\beta\gamma}} & & \Big\downarrow {\scriptstyle \mu_{\alpha\delta}} \\ P_{\gamma\delta} \otimes P_{\alpha\gamma} & \xrightarrow[\mu_{\alpha\gamma\delta}]{} & P_{\alpha\delta} \end{array}$$

is commutative.

Let us see how the class in $\mathrm{H}^3(M,\mathbb{Z})$ of a gerbe is obtained. It is often called the *Dixmier-Douady class* of the gerbe. Assume that the 2-fold intersections $U_\alpha \cap U_\beta$ are contractible, otherwise consider a refinement of the open cover and pull back all the structure of the gerbe along the refinement maps. So, the S^1-bundles $P_{\alpha\beta}$ are all trivializable. Choose sections $\sigma_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow P_{\alpha\beta}$. On a non-empty three-fold intersection, define a smooth map $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow S^1$ by

$$\mu_{\alpha\beta\gamma}(\sigma_{\beta\gamma}(x) \otimes \sigma_{\alpha\beta}(x)) = \sigma_{\alpha\gamma}(x) \cdot g_{\alpha\beta\gamma}(x).$$

Then, $g_{\alpha\beta\gamma}$ is a Čech 2-cocycle and defines a class in

$$\check{\mathrm{H}}^2(M, \underline{S^1}) \cong \mathrm{H}^3(M,\mathbb{Z}).$$

Exercise: Show that this class is independent of the possible refinement of the open cover, and of the choices of the sections $\sigma_{\alpha\beta}$.

Literature and further reading: [Mur96, Mur10]

Example 1.1.3 (The basic gerbe)**.** Let G be a compact, simple, connected, simply-connected Lie group. Upon choosing a maximal torus $T \subseteq G$, every group element $g \in G$ can be written in the form $g = h \cdot \exp(\xi) \cdot h^{-1}$ mit $\xi \in \mathfrak{t}^*$, the Lie algebra of T. A Weyl chamber is an $r := \dim(T)$-dimensional simplex $\mathfrak{A} \subseteq \mathfrak{t}^*$ such that $\xi \in \mathfrak{A}$ is uniquely determined, i.e. there is a map

$$q : G \longrightarrow \mathfrak{A} : g \longmapsto \xi.$$

Let $\nu_0, ..., \nu_r$ be the vertices of $\mathfrak{A}$. Then, G is covered by open sets $U_\alpha := q^{-1}(\mathfrak{A} \setminus f(\alpha))$, for $\alpha = 0, ..., r$, where $f(\alpha)$ is the face opposite to the vertex ν_α. It can be shown that each 2-fold intersection $U_\alpha \cap U_\beta$ can be identified with the coadjoint orbit in $\mathfrak{g}^*$ going through $\nu_{\alpha\beta} := \nu_\beta - \nu_\alpha$. In good cases this coadjoint orbit is quantizable, for example if $G = \mathrm{SU}(n)$. In these cases we find over the prequantum S^1-bundle $P_{\alpha\beta}$ over $U_\alpha \cap U_\beta$. The equation

$$\nu_{\beta\gamma} + \nu_{\alpha\beta} = \nu_{\alpha\gamma}$$

induces the isomorphism $\mu_{\alpha\beta\gamma}$. The basic gerbe represents a generator of $\mathrm{H}^3(G,\mathbb{Z}) = \mathbb{Z}$.

Literature and further reading: [GR02, Mei02, SW10]

We proceed with a generalization of the definition of a gerbe. The first generalization is to replace open covers by surjective submersions in the sense that we allow the surjective submersion

$$\coprod_{\alpha \in A} U_\alpha \longrightarrow M$$

to be any surjective submersion $\pi : Y \longrightarrow M$. The intersections of open sets can then be reproduced by fibre products:

$$\coprod_{(\alpha_1, ... \alpha_k) \in A^k} U_{\alpha_1} \cap ... \cap U_{\alpha_k} = Y \times_M .... \times_M Y =: Y^{[k]}.$$

Thus, an S^1-gerbe over M is now a surjective submersion $\pi : Y \longrightarrow M$, an S^1-bundle P over $Y^{[2]}$, and a bundle gerbe product, i.e. bundle isomorphism

$$\mu : \mathsf{pr}_{23}^* P \otimes \mathsf{pr}_{12}^* P \longrightarrow \mathsf{pr}_{13}^* P$$

which is associative over $Y^{[4]}$. The second generalization is to allow an arbitrary abelian Lie group A instead of S^1. This has the effect that instead of $\mathrm{H}^3(M, \mathbb{Z})$ only the Čech cohomology $\check{\mathrm{H}}^2(M, \underline{A})$ is relevant.

Example 1.1.4 (Lifting gerbe)**.**

(i) A central extension

$$1 \longrightarrow A \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

of Lie groups is the same as an A-bundle $P := \hat{G}$ over G which is multiplicative in the sense that it is equipped with an associative bundle morphism

$$\phi : \mathsf{pr}_2^* P \otimes \mathsf{pr}_1^* P \longrightarrow m^* P$$

over $G \times G$, where $\mathsf{pr}_1, \mathsf{pr}_2, m : G \times G \longrightarrow G$ are the projections and the multiplication. Fibrewise over a point $(g_1, g_2) \in G \times G$, this is

$$\phi_{g_1, g_2} : P_{g_2} \otimes P_{g_1} \longrightarrow P_{g_1 g_2} : (\hat{g}_2, \hat{g}_1) \longmapsto (\hat{g}_1 \cdot \hat{g}_2),$$

and the associativity condition is that the diagram

$$\begin{array}{ccc} P_{g_3} \otimes P_{g_2} \otimes P_{g_1} & \xrightarrow{\mathrm{id} \otimes \phi_{g_1, g_2}} & P_{g_3} \otimes P_{g_1 g_2} \\ \Big\downarrow{\scriptstyle \phi_{g_2, g_3} \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \phi_{g_1 g_2, g_3}} \\ P_{g_2 g_3} \otimes P_{g_1} & \xrightarrow[\phi_{g_1, g_2 g_3}]{} & P_{g_1 g_2 g_3} \end{array}$$

is commutative.

Exercise: Confirm in detail that this defines an equivalence between the category of central Lie group extensions of G by A, and the category of multiplicative A-bundles over G.

(ii) Suppose a central extension is given, and additionally a G-bundle Q over a manifold M. The question is, whether or not Q admits a $\hat{G}$-lift, i.e. a $\hat{G}$-bundle $\hat{Q}$ over M such that $p_*(\hat{Q}) = Q$, where $p : \hat{G} \longrightarrow G$. This is equivalent to asking for a map $f : \hat{Q} \longrightarrow Q$ that commutes with the projections to M, and is equivariant in the sense that

$$f(\hat{p} \cdot \hat{g}) = f(\hat{p}) \cdot p(\hat{g}).$$

It is easy to see that the answer can be given as follows. Choose a cover $\{U_\alpha\}$ of M by open sets such that it admit sections into Q, and such that the associated Čech cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ lifts along p to maps

$$\hat{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \hat{G}.$$

These will in general not be a cocycle, instead,

$$f_{\alpha\beta\gamma} := \hat{g}_{\beta\gamma} \cdot \hat{g}_{\alpha\beta} \cdot \hat{g}_{\alpha\gamma}^{-1} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow A$$

is a Čech 2-cocycle defining a class

$$[f_{\alpha\beta\gamma}] \in \check{\mathrm{H}}^2(M, \underline{A}).$$

This class vanishes if and only if the $\hat{G}$-lift $\hat{Q}$ exists.

(iii) Sadly, the construction of the obstruction class $[f_{\alpha\beta\gamma}]$ requires a lot of highly non-canonical choices. Gerbes can do better. Consider the following A-gerbe consisting of

(1) The surjective submersion $\pi : Q \longrightarrow M$ (the bundle projection).

(2) There is a canonical map

$$g : Q \times_M Q \longrightarrow G$$

defined such that $q' = q \cdot g(q, q')$. Now consider the extension an A-bundle $\hat{G}$, and take $P := g^*\hat{G}$ in the construction of the bundle gerbe.

(3) The bundle gerbe product is obtained by pulling back the multiplicative structure ϕ along the map

$$g' : Q \times_M Q \times_M Q \longrightarrow G \times G$$

with $(q', q'') = (q, q') \cdot g'(q, q', q'')$.

Exercise: Check that this is a bundle gerbe product, and show that the class of this bundle gerbe is the obstruction class $[f_{\alpha\beta\gamma}]$.

1.2 Higher algebra of gerbes

We have started by motivating how to get the definition of a gerbe from a given notion of S^1-bundles, tensor products, pullbacks of S^1-bundles and bundle morphisms, that is: from the monoidal presheaf of S^1-bundles.

We want to formalize this motivational definition. We will split it into two processes: delooping and sheafification. Delooping is easy: if $\mathcal{F}$ is a monoidal presheaf of categories, we get a presheaf $B\mathcal{F}$ of bicategories defined by

$$(B\mathcal{F})(M) := B(\mathcal{F}(M))$$

where the B on the right hand side denotes the usual construction of a bicategory with one object from of a monoidal category. In the case of the monoidal sheaf $\mathcal{B}un_{S^1}$, this is the transition from S^1-bundles to (trivial) S^1-gerbes. In general, even if $\mathcal{F}$ was a *sheaf* of categories (e.g. with respect to the Grothendieck topology of surjective submersions), then $B\mathcal{F}$ is not necessarily a sheaf anymore. This is why we need sheafification.

Recall that any presheaf $\mathcal{F}$ over the category $\mathcal{M}an$ of manifolds extends canonically to a presheaf $\mathcal{F}'$ on the category of simplicial manifolds, in particular, via the nerve construction, to the category $\mathcal{G}rpd$ of Lie groupoids (with morphisms the smooth functors), such that the restriction of $\mathcal{F}'$ to Lie groupoids with only identity morphisms satisfies

$$\mathcal{F}'|_{\mathcal{M}an} = \mathcal{F}.$$

For example, if $\mathcal{F}$ is a presheaf of categories, the presheaf $\mathcal{F}'$ assigns to a Lie groupoid Γ the following category $\mathcal{F}'(\Gamma)$:

- The objects are pairs (X, f) with $X \in \mathcal{F}(\Gamma_0)$ and $f : s^*X \longrightarrow t^*X$ is an isomorphism in $\mathcal{F}(\Gamma_1)$ such that, over the manifold $\Gamma_1 {}_s\times_t \Gamma_1$ of composable morphisms, we have

$$\mathsf{pr}_2^* f \circ \mathsf{pr}_1^* f = c^* f,$$

where c is the composition map.

- A morphism from (X, f) to (Y, g) is a morphism $h : X \longrightarrow Y$ in $\mathcal{F}(\Gamma_0)$ such that the diagram

$$\begin{array}{ccc} s^*X & \xrightarrow{f} & t^*X \\ {\scriptstyle s^*h}\downarrow & & \downarrow{\scriptstyle t^*h} \\ s^*Y & \xrightarrow[g]{} & t^*Y \end{array}$$

is commutative.

Exercise: Repeat this construction in detail for a presheaf of bicategories. Hint: the objects of the bicategory $\mathcal{F}'(\Gamma)$ should involve a condition for a 2-isomorphism over the manifold of triples of composable morphisms.

Example 1.2.1. If G acts smoothly on M we form the action groupoid $M/\!\!/G$, which is supposed to replace the quotient M/G which is often not nicely behaved. Then, $\mathcal{B}un_A(M/\!\!/G)$ is the sheaf of G-equivariant A-bundles.

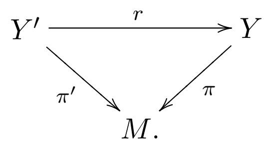
Exercise: We write $BG := pt/\!\!/G$. Show that $\mathcal{B}un_{S^1}(BG)$ is the category of Lie group homomorphisms $f : G \longrightarrow S^1$, considered as a discrete category. What are the categories $\mathcal{B}un_G(BG)$ and $\mathcal{B}un_G(BS^1)$?

Example 1.2.2. Let $\pi : Y \longrightarrow M$ be a surjective submersion. Then we have a Lie groupoid Y^* which generalizes the Čech groupoid of an open cover, i.e. the objects of Y^* are Y, and the morphisms $Y^{[2]} := Y \times_M Y$. Notice that an object in $B\mathcal{B}un_A(Y^*)$ is exactly an A-gerbe over M with surjective submersion $\pi : Y \longrightarrow M$.

The sheafification we want to perform is called the plus construction. In one line, for $\mathcal{F}$ a presheaf of bicategories, it is

$$\mathcal{F}^+(M) := \lim_{Y} \mathcal{F}(Y^*),$$

with a higher-categorical version of the direct limit (or, homotopy limit), taken along refinements of surjective submersions, i.e. diagrams

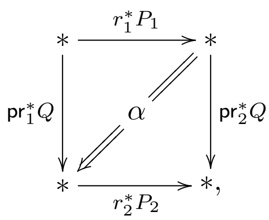


The, we have:

Definition 1.2.3. *The sheaf of A-gerbes is $\mathcal{G}rb_A := (B\mathcal{B}un_A)^+$.*

Instead of explaining the details of this higher-categorical version of a direct limit, let us spell out how it works in the case of bundle gerbes. An object in $\mathcal{G}rb_A(M)$ is a pair $(\pi : Y \longrightarrow M, X)$ consisting of a surjective submersion and an object $X \in B\mathcal{B}un_A(Y^*)$. As seen in the previous example, this is precisely an A-gerbe.

Now suppose two objects $(\pi_1 : Y_1 \longrightarrow M, X_1)$ and $(\pi_2 : Y_2 \longrightarrow M, X_2)$ in $\mathcal{G}rb_A(M)$ are given, with different surjective submersions. Then, a 1-morphism is a common refinement $\zeta : Z \longrightarrow M$, with refinement maps $r_1 : Z \longrightarrow Y_1$ and $r_2 : Z \longrightarrow Y_2$, and a 1-morphism in $B\mathcal{B}un_A(Z^*)$ from $r_1^*X_1$ to $r_2^*X_2$. If, for $i = 1, 2$, $X_i = (P_i, \mu_i)$ with A-bundles P_i over $Y_i^{[k]}$ and μ_i the bundle gerbe products, this is an A-bundle Q over Z together with a 2-isomorphism



satisfying a coherence condition over pairs of composable morphisms in Z^*, i.e. over $Z^{[2]} \times_Z Z^{[2]} = Z^{[3]}$. Translating into ordinary language, this is a bundle morphism

$$\alpha : \mathsf{pr}_2^*Q \otimes r_1^*P_1 \longrightarrow r_2^*P_2 \otimes \mathsf{pr}_1^*Q$$

satisfying a condition over $Z^{[3]}$. This given, in direct and systematical way, the correct notion of a 1-morphism between gerbes.

Exercise: Work out this condition over $Z^{[3]}$, and work out what a 2-morphism is.

Literature and further reading: [MS00, NS11, NWa]

Lemma 1.2.4. *The following statements follow from this new definition of an A-gerbe.*

(i) A-gerbes form a sheaf of monoidal bicategories over manifolds.

(ii) $\mathrm{Aut}(\mathcal{G}) = \mathcal{B}un_A(M)$ *as monoidal categories; for any bundle gerbe $\mathcal{G}$ over M.*

Example 1.2.5. The trivial bundle gerbe is given by the identity surjective submersion $\pi = \mathrm{id}_M$, so that all fibre products are again M, the trivial A-bundle $P := M \times A$, and the identity bundle morphism. A 1-isomorphism

$$\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{I}$$

is called trivialization. Choosing the surjective submersion $\pi : Y \longrightarrow M$ of $\mathcal{G}$ as a common refinement, such a trivialization consists of an A-bundle Q over Y together with an isomorphism

$$\alpha : \mathsf{pr}_2^* Q \otimes P \longrightarrow \mathsf{pr}_1^* Q \quad , \quad \alpha_{y_1,y_2} : Q_{y_2} \otimes P_{y_1,y_2} \longrightarrow Q_{y_1}$$

over $Y^{[2]}$, such that for $(y_1, y_2, y_3) \in Y^{[3]}$ the diagram

$$\begin{array}{ccc} Q_{y_3} \otimes P_{y_2,y_3} \otimes P_{y_1,y_2} & \xrightarrow{\alpha_{y_2,y_3} \otimes \mathrm{id}} & Q_{y_2} \otimes P_{y_1,y_2} \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes \mu_{y_1,y_2,y_3}} & & \Big\downarrow{\scriptstyle \alpha_{y_1,y_2}} \\ Q_{y_3} \otimes P_{y_1,y_3} & \xrightarrow[\alpha_{y_1,y_3}]{} & Q_{y_1} \end{array}$$

is commutative.

Exercise: Let $\mathcal{L}$ be the lifting gerbe for the problem of lifting the structure group of a G-bundle P to a central extension $\hat{G}$. Show that the category of $\hat{G}$-lifts of P is canonically equivalent to the category $\mathcal{H}om(\mathcal{L}, \mathcal{I})$ of trivializations of $\mathcal{L}$.

Exercise: Show that an A-gerbe over $pt /\!\!/ G$ is the same as a central Lie group extension of G by A.

Exercise: Define A-2-gerbes by

$$2\text{-}\mathcal{G}rb_A := (B\mathcal{G}rb_A)^+$$

and work out what exactly that is. Show that an A-2-gerbe over BG is a multiplicative gerbe over G.

Literature and further reading: [Ste04, Wal10a]

2 Lecture II: Twisted vector bundles with connections

2.1 Twisted vector bundles

Instead of the monoidal sheaf $\mathcal{B}un_{S^1}$ we could have taken the monoidal sheaf $\mathcal{B}un_{\mathbb{C}}$ of hermitian vector bundles (w.r.t. the tensor product, and with linear bundle isometries as morphisms). We get another 2-stack of "vector bundle gerbes",

$$\mathcal{G}rb_{\mathbb{C}} := (B\mathcal{B}un_{\mathbb{C}})^{+}.$$

We denote by $\mathcal{V}ect_{\mathbb{C}}$ the monoidal category of hermitian vector spaces. Note that the functor

$$BS^1 \longrightarrow \mathcal{V}ect_{\mathbb{C}} : pt \longmapsto \mathbb{C}$$

induces a monoidal functor

$$\mathcal{B}un_{S^1} \longrightarrow \mathcal{B}un_{\mathbb{C}}$$

and in turn another monoidal 2-functor

$$\mathcal{G}rb_{S^1} \longrightarrow \mathcal{G}rb_{\mathbb{C}}.$$

This 2-functor becomes an equivalence after groupoidification of the right hand side, i.e. after discarding all *non-invertible* 1-morphisms and 2-morphisms in $\mathcal{G}rb_{\mathbb{C}}$. This is due to the fact that the existence of the bundle gerbe product

$$\mu : \mathsf{pr}_{23}^*L \otimes \mathsf{pr}_{12}^*L \longrightarrow \mathsf{pr}_{13}^*L$$

of a gerbe in $\mathcal{G}rb_{\mathbb{C}}(M)$, which is by design an *iso*morphism, requires that L has rank one.

Exercise: Show that a 1-morphism $\mathcal{E} : \mathcal{I} \longrightarrow \mathcal{I}$ is just a hermitian vector bundle.

Definition 2.1.1. *A $\mathcal{G}$-twisted vector bundle $\mathcal{E}$ is a 1-morphism*

$$\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}$$

in $\mathcal{G}rb_{\mathbb{C}}(M)$, that is, a vector bundle E over Y together with a linear bundle isometry

$$\rho : \mathsf{pr}_2^*E \otimes L \longrightarrow \mathsf{pr}_1^*E$$

over $Y^{[2]}$, such that the diagram

$$\begin{array}{ccc} \mathsf{pr}_3^*E \otimes \mathsf{pr}_{23}^*L \otimes \mathsf{pr}_{12}^*L & \xrightarrow{\mathrm{id}\otimes\mu} & \mathsf{pr}_3^*E \otimes \mathsf{pr}_{13}^*L \\ {\scriptstyle \mathsf{pr}_{23}^*\rho\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_{13}^*\rho} \\ \mathsf{pr}_2^*E \otimes \mathsf{pr}_{12}^*L & \xrightarrow[\mathsf{pr}_{12}^*\rho]{} & \mathsf{pr}_1^*E \end{array}$$

of bundle morphisms over $Y^{[3]}$ is commutative.

A $\mathcal{G}$-twisted *line* bundle is the same as a trivialization of $\mathcal{G}$. The problem is that if E is a $\mathcal{G}$-twisted vector bundle of rank n, then $\Lambda^n E$ is a $\mathcal{G}^{\otimes n}$-twisted line bundle. In particular, $\mathcal{G}$-twisted vector bundles only exist if $\mathcal{G}$ is torsion.

$\mathcal{G}$-twisted vector bundles form a monoidal category $\mathcal{B}un_{\mathbb{C}}(M, \mathcal{G})$ under the direct sum of twisted vector bundles. If $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$ is an isomorphism between bundle gerbes over M, there is an induced functor

$$\mathcal{A}^* : \mathcal{B}un_{\mathbb{C}}(M, \mathcal{H}) \longrightarrow \mathcal{B}un_{\mathbb{C}}(M, \mathcal{G}) : \mathcal{E} \longmapsto \mathcal{E} \circ \mathcal{A}.$$

In particular, $\mathcal{B}un_{\mathbb{C}}(M, \mathcal{G})$ carries an action of the monoidal groupoid of automorphisms of $\mathcal{G}$, namely $\mathcal{B}un_{S^1}(M)$. Further, there is a cup product

$$\mathcal{B}un_{\mathbb{C}}(M, \mathcal{G}_1) \times \mathcal{B}un_{\mathbb{C}}(M, \mathcal{G}_2) \longrightarrow \mathcal{B}un_{\mathbb{C}}(M, \mathcal{G}_1 \otimes \mathcal{G}_2) : (\mathcal{E}_1, \mathcal{E}_2) \longmapsto \mathcal{E}_1 \otimes \mathcal{E}_2.$$

Literature and further reading: [Wal07]

The Grothendieck group of the monoid of isomorphism classes is $K(\mathrm{h}_0(\mathcal{B}un_{\mathbb{C}}(M, \mathcal{G})))$. It forms the twisted K-theory of M, just as the Grothendieck group of untwisted vector bundles form the untwisted K-theory:

Theorem 2.1.2 ([BCM+02, CW08])**.** *Let $\mathcal{G}$ be a torsion gerbe. Then, there is an isomorphism*

$$K^0(M, \mathcal{G}) \cong K(\mathrm{h}_0(\mathcal{B}un_{\mathbb{C}}(M, \mathcal{G})))$$

between the complex K-theory of M twisted by the class of $\mathcal{G}$ in $\mathrm{H}^3(M, \mathbb{Z})$, and the Grothendieck group of the monoid of isomorphism classes of $\mathcal{G}$-twisted vector bundles over M.

For twisted K-theory with non-torsion twists, one can consider $\mathcal{G}$-twisted Hilbert bundles whose structure group reduces to a certain subgroup of $U(\mathscr{H})$. A different method, which is also suitable to include odd K-theory, is the following. Let $f : Q \longrightarrow M$ be a smooth map between oriented manifolds. Consider the spin^c lifting gerbes $w_3(Q)$ and $w_3(M)$, and consider $w_3(f) := w_3(Q) \otimes f^*w_3(M)^\vee$, which is an S^1-gerbe over Q. There is a pushforward map

$$K^p(Q, f^*\mathcal{G} \otimes w_3(f)) \longrightarrow K^{p+\dim M - \dim Q}(M, \mathcal{G}).$$

Definition 2.1.3. *A $\mathcal{G}$-brane is a smooth manifold Q with a smooth map $f : Q \longrightarrow M$ and a $(f^*\mathcal{G} \otimes w_3(f))$-twisted vector bundle. A $\mathcal{G}$-brane is called Type A, if $\dim M - \dim Q$ is even, and Type B otherwise.*

Thus, $\mathcal{G}$-branes define classes in the twisted K-theory $K^d(M, \mathcal{G})$, with $d = \dim M - \dim Q$. Every class can be obtained in this way, see [Wan08].

Example 2.1.4. The map f is K-oriented if and only if $w_3(f) = 0$. In this case the pushforward just changes degree, not the twist.

Example 2.1.5. If M is a spin^c-manifold, and Q is a submanifold, then $w_3(M) = 0$, and $w_3(Q)$ is the third Stiefel Whitney class of the normal bundle of Q in M. The Freed-Witten anomaly cancellation condition is

$$f^*\mathcal{G} \otimes w_3(Q) = 0.$$

If this condition is satisfied, a $\mathcal{G}$-brane based on Q is just an ordinary vector bundle, and defines a class in $K^{\mathrm{codim} Q}(M, \mathcal{G})$.

Literature and further reading: [FW99, Wit98, Kap00, GR02, Bau09]

2.2 Connections

The main motivation for connections is that they should induce notions of parallel transport and holonomy. The trivial bundle gerbe $\mathcal{I}$ over M is supposed to have a well-defined S^1-valued holonomy around oriented, closed surfaces in M, i.e. for smooth maps $\phi : \Sigma \longrightarrow M$ with Σ a closed oriented surface. Obviously, a connection on $\mathcal{I}$ must be a 2-form $B \in \Omega^2(M)$, and the holonomy is

$$\mathrm{Hol}_{\mathcal{I}_B}(\phi) := \exp\left(\int_\Sigma \phi^* B\right) \in S^1.$$

The 3-form $H := dB \in \Omega^3(M)$ is called the curvature of $\mathcal{I}_B$. If B is an oriented 3-dimensional manifold with $\partial B = \Sigma$, and $\Phi : B \longrightarrow M$ is a smooth map with $\Phi|_\Sigma = \phi$, then we have

$$\mathrm{Hol}_{\mathcal{I}_B}(\phi) = \exp\left(\int_B \Phi^* H\right).$$

Exercise: Try to prove this. At least, make yourself clear that it does not just follow in a trivial way from Stokes' theorem, as claimed in several places in the literature!

Recall that the 1-morphisms between trivial gerbes are exactly the S^1-bundles P over M. We demand that a connection-preserving morphism between $\mathcal{I}_B$ and $\mathcal{I}_{B'}$ should imply that the two surface holonomies coincide. This is obviously the case, if the bundle P of the morphism carries a connection ω of curvature $\mathrm{curv}(\omega) = B' - B$, since the curvature 2-form has integral periods. Note that for 1-morphisms being connection-preserving is structure, not property! 2-morphisms are called connection-preserving, if they are connection-preserving in the ordinary sense.

Above we have defined bundle gerbes as the sheafification of the presheaf $B\mathcal{B}un_A$. Note that, in retrospect, $B\mathcal{B}un_A$ is the presheaf of trivial gerbes, $\mathcal{T}riv\mathcal{G}rb_A := B\mathcal{B}un_A$. Above we have defined the presheaf $\mathcal{T}riv\mathcal{G}rb^\nabla_{S^1}$ of trivial S^1-gerbes with connection, and so it is clear what to do:

Definition 2.2.1. *The monoidal sheaf of bundle gerbes with connection is*

$$\mathcal{G}rb^\nabla_A := (\mathcal{T}riv\mathcal{G}rb^\nabla_A)^+.$$

Splitting the new information from the old one, we see: a connection on a bundle gerbe $\mathcal{G}$ is a 2-form $B \in \Omega^2(Y)$ together with a connection on the principal S^1-bundle P such that:

1. $\mathsf{pr}_1^* B - \mathsf{pr}_2^* B = \mathrm{curv}(P)$.
2. the bundle gerbe product μ is connection-preserving.

Exercise: Check that any connection on the trivial bundle gerbe $\mathcal{I}$ is really just a 2-form $B \in \Omega^2(M)$ (i.e. that the connection on its trivial A-bundle is automatically trivial.)

Theorem 2.2.2 ([Mur96])**.** *Every bundle gerbe admits a connection.*

Since the curvature $H := dB$ of $\mathcal{I}_B$ is invariant under 1-morphisms, it follows that it survives the sheafification, so that every bundle gerbe with connection has a well-defined curvature

$$\mathrm{curv}(\mathcal{G}) \in \Omega^3(M).$$

Exercise: Check this manually!

Literature and further reading: [Mur96, Wal10b, FNSW08]

Let us see how the surface holonomy for an S^1-bundle gerbe $\mathcal{G}$ with connection is defined. Suppose $\phi : \Sigma \longrightarrow M$ is a closed oriented surface in M. Then, since $\mathrm{H}^3(\Sigma, Z) = 0$, $\phi^*\mathcal{G}$ admits a trivialization $\mathcal{T} : \phi^*\mathcal{G} \longrightarrow \mathcal{I}$. It pulls back the connection on $\mathcal{G}$ to a connection on $\mathcal{I}$, i.e. to a 2-form $B \in \Omega^2(\Sigma)$, such that $\mathcal{T} : \phi^*\mathcal{G} \longrightarrow \mathcal{I}_B$ is connection-preserving. Then, define

$$\mathrm{Hol}_{\mathcal{G}}(\phi) := \exp\left(\int_\Sigma B\right).$$

Exercise: Check that this does not depend on the choices of B and $\mathcal{T}$, by using that two trivialization compose to an endomorphism of trivial gerbes.

In their applications to string theory, S^1-gerbes with connection are the B-fields, and their surface holonomy if the contribution to the action functional of the string.

Literature and further reading: [Gaw88, CM86, FNSW08]

Remark 2.2.3. Surface holonomy can be defined for A-gerbes, with a general abelian Lie group A. However, above definition does not work: if, for example, the structure is $A = \mathbb{Z}_2$, the the gerbe $\phi^*\mathcal{G}$ is classified by $\mathrm{H}^2(\Sigma, \mathbb{Z}_2)$, which might not vanish. See Section 3.3 of [Wal].

Example 2.2.4 (Connections on lifting gerbes)**.** Recall that for a central extension

$$1 \longrightarrow A \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

and a G-bundle P over M we have constructed a lifting bundle gerbe $\mathcal{L}_P$ with the property that

$$\mathcal{H}om(\mathcal{L}_P, \mathcal{I}) \cong \hat{G}\text{-}\mathcal{L}ift(P),$$

i.e. the trivialization of $\mathcal{L}_P$ are exactly the $\hat{G}$-lifts of P. Now assume that P carries a connection, and assume two additional structure called split and reduction. Then, the lifting bundle gerbe $\mathcal{L}_P$ can be equipped with a connection constructed. The reduction can be used to assign an $\mathfrak{a}$-valued scalar curvature to any $\hat{G}$-lift $\hat{P}$, and we let $\hat{G}\text{-}\mathcal{L}\mathit{ift}^{\nabla}_{\rho}(P)$ denote the category of $\hat{G}$-lifts with fixed scalar curvature ρ. Then, one can show that one gets an equivalence of categories:

$$\mathcal{H}om^{\nabla}(\mathcal{L}_P, \mathcal{I}_\rho) \cong \hat{G}\text{-}\mathcal{L}\mathit{ift}^{\nabla}_{\rho}(P).$$

Literature and further reading: [Gom03, Wal11]

Recall that a $\mathcal{G}$-twisted vector bundle is a 1-morphism $\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}$ in $\mathcal{G}rb_{\mathbb{C}}(M)$. Now we consider the presheaf $\mathcal{T}riv\mathcal{G}rb^{\nabla}_{\mathbb{C}}$ with objects the 2-forms, 1-morphisms from B to B' the vector bundles E with connection such that

$$B - B' = \frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)),$$

where n is the rank of E. If $\mathcal{G}$ is equipped with a connection we can look at connection-preserving 1-morphisms

$$\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}_F,$$

for $F \in \Omega^2(M)$. This is a connection on the vector bundle E over Y, such that

$$B - \pi^* F = \frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)),$$

and the bundle morphism

$$\rho : \mathsf{pr}_2^* E \otimes L \longrightarrow \mathsf{pr}_1^* E$$

is connection-preserving. The form F is called the curvature of the connection on the $\mathcal{G}$-twisted vector bundle $\mathcal{E}$. It is in fact determined by the connection on E. Note that $dF = H$, with H the curvature of the gerbe.

Example 2.2.5. Recall that if $\mathcal{I}$ is the trivial bundle gerbe, an $\mathcal{I}$-twisted vector bundle is the same as a vector bundle. If $B \in \Omega^2(M)$ is a 2-form, then an $\mathcal{I}_B$-twisted vector bundle with connection is just a vector bundle with connection (although one could have guessed that it is not, for it still seems to be twisted by pure differential form data). The curvature is

$$F = B + \frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)) \in \Omega^2(M).$$

If $(Q, i, \mathcal{E})$ is a $\mathcal{G}$-brane, then a connection on $\mathcal{E}$ is called Chan-Patton field in the physics literature. Suppose Σ is an oriented compact surface with boundary, and $\phi : \Sigma \longrightarrow M$ is a smooth map together with a lift of $\phi|_{\partial\Sigma}$ through Q, i.e. a smooth map $\phi_Q : \partial\Sigma \longrightarrow Q$ such that $i \circ \phi_Q = \phi|_{\partial\Sigma}$. Let $\mathcal{T} : \phi^*\mathcal{G} \longrightarrow \mathcal{I}_B$ be a connection-preserving trivialization. Then we have a 1-morphism

$$\mathcal{I}_B|_{\partial\Sigma} \xrightarrow{\mathcal{T}^{-1}} \phi^*\mathcal{G}|_{\partial\Sigma} \xrightarrow{\phi_Q^*\mathcal{E}} \mathcal{I}_{\phi_Q^* F},$$

i.e. a vector bundle E with connection over $\partial\Sigma$ of (traced) curvature $\phi^*F - B$, and

$$\mathrm{Hol}_{\mathcal{G},\mathcal{E}}(\phi) := \exp\left(\int_\Sigma B\right) \cdot \mathrm{tr}(\mathrm{Hol}_E(\partial\Sigma))$$

is well-defined.

Literature and further reading: [CJM02, KL04, Wal07]

3 Lecture III: Non-abelian gerbes and twistings

3.1 Generalization of structure groups

The following table shows, in full generality, what the structure group of an n-gerbe may be (for $n = -1, 0, 1, 2$).

functions	bundles	gerbes	2-gerbes
set	group	abelian group	abelian group
	groupoid	2-group	braided 2-group
		2-groupoid	3-group
			3-groupoid

In this table:

- going to the left means: adding group structure / commutativity
- going up means: taking the thing with just one object

Notice that usually one only considers the first row: functions with values in a *set*, bundles with structure *groups*, gerbes with *abelian structure groups*, and so on. However, it is completely natural, and in fact useful, to allow the more general things in the rows below.

We start to explain groupoid bundles, which appear in the second column. That's worthwhile because they will be used to define 2-group-gerbes. Let Γ be a Lie groupoid. We say that a *right action* of a Lie groupoid Γ on a smooth manifold M is a pair (ϕ, ρ) consisting of smooth maps $\phi : M \longrightarrow \Gamma_0$ and $\rho : M \,{}_{\alpha}\!\times_t \Gamma_1 \longrightarrow M$ such that

$$\rho(\rho(x, g), h) = \rho(x, g \circ h) \quad , \quad \rho(x, \mathrm{id}_{\phi(x)}) = x \quad \text{and} \quad \phi(\rho(x, g)) = s(g)$$

for all possible $g, h \in \Gamma_1$, $p \in \Gamma_0$ and $x \in M$. The map ϕ is called *anchor*.

Definition 3.1.1. *A* <u>*Γ-bundle over M*</u> *is a smooth manifold P with a surjective submersion $\pi : P \longrightarrow M$ and a right Γ-action (ϕ, ρ) that respects the projection π, such that*

$$\tau : P \,{}_{\phi}\!\times_t \Gamma_1 \longrightarrow P \times_M P : (p, g) \longmapsto (p, \rho(p, g))$$

is a diffeomorphism.

Γ-bundles over M form a category $\mathcal{B}un_\Gamma(M)$.

Example 3.1.2.

(i) For X a smooth manifold considered as a Lie groupoid with only identity morphisms, we have an equivalence of categories

$$\mathcal{B}un_X(M) \cong C^\infty(M, X).$$

(ii) For G a Lie group and $\Gamma = BG$ the Lie groupoid with just one object,

$$\mathcal{B}un_\Gamma(M) \cong \mathcal{B}un_G(M),$$

with ordinary G-bundles on the right hand side (note that the notation is not very good).

(iii) For $\Gamma = X /\!/ H$ an action groupoid, a Γ-bundle is the same as an ordinary H-bundle together with a smooth H-anti-equivariant map $\phi : P \longrightarrow X$, i.e.

$$\phi(p \cdot h) = h^{-1} \cdot \phi(x).$$

A morphism between $X /\!/ H$-bundles is the same as an ordinary H-bundle morphism that exchanges the anti-equivariant maps to X.

Exercise: Proof the equivalences of the previous example!

In the following we restrict our attention to action groupoids, in particular because all 2-groups are action groupoids. Then we use only the equivalent description of (iii) in the previous example.

Just like ordinary bundles, one can push groupoid bundles along functors $F : \Gamma \longrightarrow \Omega$ of their structure Lie groupoids in terms of a functor

$$\mathcal{B}un_\Gamma \longrightarrow \mathcal{B}un_\Omega.$$

Let H act on X and H' act on X'. Consider a group homomorphism $\varphi : H \longrightarrow H'$ and a smooth map $f : X \longrightarrow X'$ such that

$$f(hx) = \varphi(h)f(x).$$

Then, $F := (\varphi \times f, f) : X/\!\!/H \longrightarrow X'/\!\!/H'$ is a functor between the action groupoids. If (P, ϕ) is an $X/\!\!/H$-bundle, then the bundle $F_*(P)$ has the total space $F_*(P) := P \times_H H'$, so that the elements are pairs (p, h'), and the action identifies (ph, h') with $(p, \varphi(h)h')$. It has the anchor

$$F_*(P) \longrightarrow X' : (p, h') \longmapsto h'^{-1} f(\phi(p)).$$

The following example is the prototype why considering more general structure groups is a good idea.

Example 3.1.3. Let H be a Lie group. We let H act trivially on $\mathbb{Z}_2$, and consider the action groupoid $\Gamma := \mathbb{Z}_2/\!\!/H = \mathbb{Z}_2 \times BH$. Then, a Γ-bundle is an H-bundle together with a smooth $\mathbb{Z}_2$ valued map on its base. For S^1 this is called a $\mathbb{Z}_2$-graded S^1-bundle.

Remark 3.1.4. There is a notion of connections on principal groupoid bundles, however, it turns out that it is not helpful for non-abelian gerbes. Just for completeness, we mention that for an action groupoid $X/\!\!/H$, a connection on a $X/\!\!/H$-bundle P is an ordinary connection $A \in \Omega^1(P, \mathfrak{h})$ on P satisfying

$$\rho_{A_p(\xi)}(\phi(p)) = -T_p\phi(\xi),$$

where $\phi : P \longrightarrow X$ is the H-anti-equivariant map, $p \in P$, $\xi \in T_pP$, und $\rho_X : X \longrightarrow TX$ is the vector field defined by the infinitesimal action of $X \in \mathfrak{h}$ on X,

$$\rho_X(x) := \left.\frac{d}{dt}\right|_0 \exp(tX) \cdot x.$$

Literature and further reading: [NWa, MRS, MM03, SW09]

What we want to do next is to replace the sheaf $\mathcal{B}un_A$ that we used in the definition of A-gerbes, by the sheaf $\mathcal{B}un_\Gamma$. Recall that the first step was delooping, which was relying on the monoidal structure on $\mathcal{B}un_A$. In general, $\mathcal{B}un_\Gamma$ is only monoidal for particular Lie groupoids Γ called Lie 2-groups.

Definition 3.1.5. *A (strict) Lie 2-group is a Lie groupoid Γ whose objects and morphisms are Lie groups, and all whose structure maps are Lie group homomorphisms.*

Example 3.1.6. Let $t : H \longrightarrow G$ be a homomorphism of Lie groups, and let $G/\!\!/H$ be the action groupoid of the H-action on G which is

$$(h, g) \longmapsto t(h)g.$$

This Lie groupoid becomes a Lie 2-group if the following structure is given: a smooth left action of G on H by Lie group homomorphisms, denoted $\alpha : G \times H \longrightarrow H$, satisfying

$$t(\alpha(g, h)) = gt(h)g^{-1} \quad \text{and} \quad \alpha(t(x)) = hxh^{-1}$$

for all $g \in G$ and $h, x \in H$. Indeed, the objects G of $G/\!\!/H$ already form a Lie group, and the multiplication on the morphisms $H \times G$ of $G/\!\!/H$ is the semi-direct product

$$(h_2, g_2) \cdot (h_1, g_1) = (h_2\, \alpha(g_2, h_1), g_2 g_1). \tag{3.1.1}$$

The homomorphism $t : H \longrightarrow G$ together with the action of G on H is called a *smooth crossed module*. Summarizing, every smooth crossed module defines a Lie 2-group. One can show that every Lie 2-group arises this way.

Proposition 3.1.7. *If Γ is a Lie 2-group, then $\mathcal{B}un_\Gamma$ is a monoidal stack.*

The abstract reason for this is that one bundles the multiplications on the objects and morphisms of Γ to a multiplication functor

$$m : \Gamma \times \Gamma \longrightarrow \Gamma,$$

and then pushes the direct product of two Γ-bundles along m. If $\Gamma = G/\!\!/H$ is a Lie 2-group, and we use the reformulation of $G/\!\!/H$-bundles in terms of H-bundles with G-valued anchor, this boils down to the following construction.

Let (P, f) and (Q, g) be the principal H-bundles together with their H-anti-equivariant maps that belong to P and Q, respectively. Then, the tensor product is

$$P \otimes Q = \big(P \times_M Q\big)/\sim \qquad \text{where} \qquad (p \cdot h, q) \sim (p, q \cdot (\alpha(f(p)^{-1}, h))).$$

The action of H on $P \otimes Q$ is $[(p, q)] \cdot h = [(p \cdot h, q)]$, and the H-anti-equivariant map is $[(p, q)] \longmapsto f(p) \cdot g(q)$.

Remark 3.1.8. Every Lie 2-group as homotopy groups:

$$\pi_0\Gamma = G/t(H) \quad \text{and} \quad \pi_1\Gamma = \mathrm{Aut}(1_{\Gamma_0}) = \ker(t).$$

A Lie 2-group is called smoothly separable, if these are again Lie groups, such that the projection $\Gamma_0 \longrightarrow \pi_0\Gamma$ is a surjective submersion. Another invariant is the so-called k-invariant of Γ, which is a class in $\mathrm{H}^3(B\pi_0\Gamma, \pi_1\Gamma)$. It is represented by a multiplicative $\pi_1\Gamma$-gerbe with surjective submersion $\Gamma_0 \longrightarrow \pi_0\Gamma$, see [NWb].

Example 3.1.9. Consider the (trivial) action groupoid $\Gamma := \mathbb{Z}_2 /\!\!/ S^1$. Its homotopy groups are $\pi_0(\Gamma) = \mathbb{Z}_2$ and $\pi_1(\Gamma) = S^1$. There are two actions of $\mathbb{Z}_2$ on $\mathrm{U}(1)$, the trivial one and the one by inversion. Both give 2-groups:

1. For the trivial action it is the 2-group $\mathbb{Z}_2 \times BS^1$. The monoidal stack $\mathcal{B}un_{\mathbb{Z}_2 \times BS^1}$ consists of $\mathbb{Z}_2$-graded principal S^1-bundles (P, α). The monoidal structure is

$$(P_1, \alpha_1) \otimes (P_2, \alpha_2) = (P_1 \otimes P_2, \alpha_1\alpha_2).$$

2. For the inversion action, it is called the automorphism 2-group of S^1, $\Gamma = \mathrm{AUT}(S^1)$. The monoidal stack $\mathcal{B}un_{\mathrm{AUT}(S^1)}$ is precisely consists of $\mathbb{Z}_2$-graded principal S^1-bundles, with a twisted tensor product,

$$(P_1, \alpha_1) \otimes (P_2, \alpha_2) = (P_1 \otimes P_2^{\alpha_1}, \alpha_1\alpha_2).$$

Here, and in the following, we write

$$P^x := \begin{cases} P & x = 1 \\ P^\vee & x = -1 \end{cases}$$

with $P^\vee$ the dual bundle.

The 2-group BS^1 admits two different braidings, which are both symmetric. A braiding is a map

$$b : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathrm{U}(1).$$

The two "hexagon" axioms boil down to

$$b(x, yz) = b(x, z) \cdot b(x, y) \quad \text{and} \quad b(xy, z) = b(x, z) \cdot b(y, z).$$

This leaves two possibilities: the trivial one ($b = 1$) and

$$b(x, y) = \begin{cases} -1 & \text{if } x = y = -1 \\ 1 & \text{otherwise.} \end{cases}$$

Exercise: Show that $\mathrm{AUT}(S^1)$ does not admit any braidings (there is no natural transformation as required).

Remark 3.1.10. For every Lie group H we can let H act on $\mathrm{Aut}(H)$ by left composition with inner automorphisms. The associated action groupoid is denoted $\mathrm{AUT}(H) := H /\!\!/ \mathrm{Aut}(H)$.

- An $\mathrm{AUT}(H)$-bundle is the same as a principal H-bundle P together with a smooth, H-anti-equivariant map $\phi : P \longrightarrow \mathrm{Aut}(H)$. In the older literature, instead, one often considers H-bibundles. This is the same: the left H-action on P is defined by

  $$h \star p := p \cdot \alpha(\phi(p), h).$$

  Exercise: Check that this is a left action and commutes with the right action.

  Exercise: Show that the homotopy groups of $\mathrm{AUT}(H)$ are $\pi_0 = \mathrm{Out}(H)$ and $\pi_1 = Z(H)$.

- $\mathrm{AUT}(H)$ is in fact a Lie 2-group, which is induced by the crossed module with H and $G = \mathrm{Aut}(H)$, $t : H \longrightarrow \mathrm{Aut}(H)$ the assignment of inner automorphisms, and the action

  $$\alpha(\varphi, h) := \varphi(h).$$

  The tensor product of H-bibundles (defined by dividing out the middle action) coincides with the above defined tensor product.

Literature and further reading: [NWa]

3.2 Non-abelian gerbes

Definition 3.2.1. *Let Γ be a Lie 2-group. The sheaf of Γ-bundle gerbes is defined by*

$$\mathcal{G}rb_\Gamma := (B\mathcal{B}un_\Gamma)^+.$$

Γ-gerbes are classified by Giraud's non-abelian cohomology $\mathrm{H}^1(M,\Gamma)$. The Lie 2-group $\Gamma = BA$, for A an abelian Lie group, reproduces $\mathcal{G}rb_A$ in the sense used in Lecture II.

Example 3.2.2 (Deloopings of S^1)**.**

- A $(\mathbb{Z}_2 \times BS^1)$-bundle gerbe over M is a surjective submersion $\pi : Y \longrightarrow M$, a principal $(\mathbb{Z}_2 \times BS^1)$-bundle (P,α) over $Y^{[2]}$, and a bundle gerbe product μ over $Y^{[3]}$. Since μ is anchor-preserving, we have

  $$\alpha(y_1,y_2)\alpha(y_2,y_3) = \alpha(y_1,y_3).$$

  Thus, α represents a class in $\mathrm{H}^1(M,\mathbb{Z}_2)$. $(\mathbb{Z}_2 \times BS^1)$-bundle gerbes over M are classified by

  $$\mathrm{H}^3(M,\mathbb{Z}) \times \mathrm{H}^1(M,\mathbb{Z}_2).$$

  The homotopy groups of $B(\mathbb{Z}_2 \times BS^1)$ are $\pi_0 = 0$, $\pi_1 = \mathbb{Z}_2$ and $\pi_2 = S^1$.

- For the 2-group $\mathrm{AUT}(S^1)$, we get a surjective submersion $\pi : Y \longrightarrow M$, a $\mathbb{Z}_2$-graded S^1-bundle (P,ϕ) over $Y^{[2]}$ such that

  $$\phi(y_1,y_2)\phi(y_2,y_3) = \phi(y_1,y_3).$$

  and a bundle isomorphism

  $$\mu : \mathsf{pr}_{23}^* P \otimes \mathsf{pr}_{12}^* P^{\phi\circ \mathsf{pr}_{23}} \longrightarrow \mathsf{pr}_{13}^* P$$

  such that over $Y^{[3]}$ the diagram

  $$\begin{array}{ccc} \mathsf{pr}_{34}^*P \otimes \mathsf{pr}_{23}^*P^{\phi\circ\mathsf{pr}_{34}} \otimes \mathsf{pr}_{12}^*P^{\phi\circ\mathsf{pr}_{23}\cdot\mathsf{pr}_{34}} & \xrightarrow{\mathrm{id}\otimes\mathsf{pr}_{123}^*\mu^{\phi\circ\mathsf{pr}_{34}}} & \mathsf{pr}_{34}^*P \otimes \mathsf{pr}_{13}^*P^{\phi\circ\mathsf{pr}_{34}} \\ \downarrow{\scriptstyle \mathsf{pr}_{234}^*\mu\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathsf{pr}_{134}^*\mu} \\ \mathsf{pr}_{24}^*P \otimes \mathsf{pr}_{12}^*P^{\phi\circ\mathsf{pr}_{24}} & \xrightarrow[\mathsf{pr}_{124}^*\mu]{} & \mathsf{pr}_{14}^*P \end{array}$$

  is commutative. We see that $\mathrm{AUT}(S^1)$-gerbes are classified by a group (namely, Giraud's non-abelian cohomology group) that sits in a sequence

  $$\ldots \longrightarrow \mathrm{H}^3(M,\mathbb{Z}) \longrightarrow \mathrm{H}^1(M,\mathrm{AUT}(S^1)) \longrightarrow \mathrm{H}^1(M,\mathbb{Z}_2) \longrightarrow \ldots$$

- One can also look at a mixture of both 2-groups, namely at one which comes from the crossed module $G := \mathbb{Z}_2 \times \mathbb{Z}_2$, $H = S^1$, $t : H \longrightarrow G$ is constant as before, and the G action on H is induced from the inversion action along the projection $\mathsf{pr}_1 : G \longrightarrow \mathbb{Z}_2$. This is the 2-group $\mathrm{AUT}(S^1) \times \mathbb{Z}_2$. An $(\mathrm{AUT}(S^1) \times \mathbb{Z}_2)$-gerbe is an S^1-bundle P over $Y^{[2]}$ and a map $Y^{[2]} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$, which we write as (ϕ, c). The bundle isomorphism μ as well as the diagram as as before. The map c will only be involved when discussing twisted vector bundles in the following example.

Literature and further reading: [Gir71, Bre90, ACJ05, NWa]

Example 3.2.3 (Twisted vector bundles)**.** We denote by $\mathcal{V}ect_{gr\mathbb{C}}$ the monoidal category of $\mathbb{Z}_2$-graded hermitian vector spaces, with morphisms the graded linear isometries, and by $\mathcal{B}un_{gr\mathbb{C}}$ the associated monoidal stack of $\mathbb{Z}_2$-graded hermitian vector bundles.

(i) Consider the functor

$$\mathbb{Z}_2 \times BS^1 \longrightarrow \mathcal{V}ect_{gr\mathbb{C}}$$

which sends an object $x \in \mathbb{Z}_2$ to $\mathbb{C}$, considered as odd or even depending on x, and which sends a morphism $(z, x) : x \longmapsto x$ to the graded linear isometry which is multiplication by z. This is a monoidal functor with respect to the graded tensor product. It induces a 2-functor

$$\mathcal{G}rb_{\mathbb{Z}_2 \times BS^1} \longrightarrow \mathcal{G}rb_{gr\mathbb{C}} := (B\mathcal{B}un_{gr\mathbb{C}})^+.$$

If $\mathcal{G}$ is a $(\mathbb{Z}_2 \times BS^1)$-gerbe, then a $\mathcal{G}$-module is a 1-morphism $\mathcal{E} : \mathcal{G} \longrightarrow \mathcal{I}$ in $\mathcal{G}rb_{gr\mathbb{C}}$. That is: a $\mathbb{Z}_2$-graded hermitian vector bundle E over Y together with a graded linear bundle isometry

$$\rho : \mathsf{pr}_2^* E \otimes L \longrightarrow \mathsf{pr}_1^* E.$$

Note that the tensor product on the left hand side is a tensor product of graded vector bundles, where L is graded according to the map $c : Y^{[2]} \longrightarrow \mathbb{Z}_2$. So, if for $(y_1, y_2) \in Y^{[2]}$ we have $c(y_1, y_2) = 1$, then ρ_{y_1,y_2} splits as

$$\rho^+_{y_1,y_2} : E^+_{y_2} \otimes L \longrightarrow E^+_{y_1} \quad \text{and} \quad \rho^-_{y_1,y_2} : E^-_{y_2} \otimes L \longrightarrow E^-_{y_1}.$$

If $c(y_1, y_2) = -1$, then ρ_{y_1,y_2} splits as

$$\rho^+_{y_1,y_2} : E^-_{y_2} \otimes L \longrightarrow E^+_{y_1} \quad \text{and} \quad \rho^-_{y_1,y_2} : E^+_{y_2} \otimes L \longrightarrow E^-_{y_1}.$$

Thus we might say that ρ_{y_1,y_2} is even or odd depending on $c(y_1, y_2)$.

(ii) Next we want to replace BS^1 by $\mathrm{AUT}(S^1)$ in the previous example.

Exercise: Show that it is not possible to construct a monoidal functor

$$\mathrm{AUT}(S^1) \longrightarrow \mathcal{V}ect_{gr\mathbb{C}}$$

Instead we use ungraded hermitian vector spaces and send the $\mathbb{Z}_2$-factor into a separate artificial $\mathbb{Z}_2$-factor, i.e. we use category

$$\mathcal{V}ect_{\mathbb{C},\mathbb{Z}_2}$$

whose objects are pairs (V,x) of a hermitian vector space V and of a sign $x \in \mathbb{Z}_2$, and whose morphisms from (V,x) to (V',x') are linear maps $f : V^{xx'} \longrightarrow V'$, where $V^{-1} := \overline{V}$ is the opposed vector space and $V^1 := V$. It is equipped with a monoidal structure given on objects by

$$(V_1,x_1) \otimes (V_2,x_2) := (V_1 \otimes V_2^{x_1}, x_1x_2),$$

On morphisms, we put

$$\begin{aligned}(f_1 : (V_1,x_1) \longrightarrow (V_1',x_1')) \otimes (f_2 : (V_2,x_2) \longrightarrow (V_2',x_2'))\\ = (f_1^{x_2x_2'} \otimes f_2 : (V_1 \otimes V_2^{x_1}) \longrightarrow (V_1' \otimes V_2'^{x_1'})),\end{aligned}$$

where f^x denotes the map f followed by complex conjugation (recall that if $f : V_1 \longrightarrow V_2$ is $\mathbb{C}$-linear, than the same map $f : \overline{V}_1 \longrightarrow \overline{V}_2$ is also $\mathbb{C}$-linear, and the maps $f^{-1} : V_1 \longrightarrow \overline{V_2}$ and $f^{-1} : \overline{V_1} \longrightarrow V_2$ are also both $\mathbb{C}$-linear). Now we look at the functor

$$F : \mathrm{AUT}(S^1) \longrightarrow \mathcal{V}ect_{\mathbb{C},\mathbb{Z}_2}$$

which sends an object x to $(\mathbb{C},x)$ and a morphism $(z,x) : x \longrightarrow x$ to the linear map which is multiplication by z.

Exercise: Show that this functor is strongly monoidal, i.e. there is a natural equivalence $\otimes \circ (F \times F) \cong F \circ \otimes$.

Hint: Find first that the structure of such a natural equivalence is, for each pair $(x_1,x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, a linear isomorphism $\mathbb{C} \otimes \mathbb{C}^{x_1} \cong \mathbb{C}$. Show that these can be chosen such that the naturality condition is satisfied.

Now suppose that $\mathcal{G}$ is an $\mathrm{AUT}(S^1)$-gerbe over M. A $\mathcal{G}$-twisted vector bundle in this context is now a hermitian vector bundle E over Y together with a map $e : Y \longrightarrow \mathbb{Z}_2$ and vector bundle isometry

$$\rho_{y_1,y_2} : \left(E_{y_2} \otimes L_{y_1,y_2}^{e(y_2)}\right)^{\phi(y_1,y_2)e(y_1)e(y_2)} \longrightarrow E_{y_1}$$

satisfying a condition over $Y^{[3]}$. Note that there is no condition of the form $\phi(y_1,y_2)e(y_2) = e(y_1)$, since the category $\mathcal{V}ect_{\mathbb{C},\mathbb{Z}_2}$ has morphisms between (V,x) and (V',x') for arbitrary x,x'.

Exercise: Derive explicitly the commutative diagram over $Y^{[3]}$ that describes the condition for ρ.

(iii) Finally, let us combine the previous two examples, and bundle them into the functor

$$F : \mathrm{AUT}(S^1) \times \mathbb{Z}_2 \longrightarrow \mathcal{V}ect_{gr\mathbb{C},\mathbb{Z}_2}.$$

Recall that the first $\mathbb{Z}_2$-factor – the one of $\mathrm{AUT}(S^1)$ – goes to the artificial $\mathbb{Z}_2$-factor on the right hand side, whereas the second $\mathbb{Z}_2$-factor – the direct factor – dictates the grading of the vector spaces.

Now a twisted vector bundle is a $\mathbb{Z}_2$-graded hermitian vector bundle E over Y together with a map $e : Y \longrightarrow \mathbb{Z}_2$, and a graded linear bundle isometry

$$\rho_{y_1,y_2} : (E_{y_2} \otimes L_{y_1,y_2}^{e(y_2)})^{\phi(y_1,y_2)e(y_1)e(y_2)} \longrightarrow E_{y_1}$$

with the grading of L_{y_1,y_2} determined by $c(y_1,y_2)$.

Example 3.2.4 (An equivariant example)**.** We consider an $\mathrm{AUT}(S^1)$-gerbe over BG. We can assume that the gerbe over the point is just trivial, so that we have to spell out what a G-equivariant structure on the trivial $\mathrm{AUT}(S^1)$-gerbe is. By definition, it is an isomorphism $\mathcal{A}$ between trivial $\mathrm{AUT}(S^1)$-gerbes over G, i.e. an S^1-bundle P over G together with a map $\phi : G \longrightarrow \mathbb{Z}_2$. We switch to the point-wise notation, and denote by P_g the fibre of P over $g \in G$. The further structure consists of a 2-isomorphism over $G \times G$, i.e. an isomorphism

$$\lambda_{g_1,g_2} : P_{g_2} \otimes P_{g_1}^{\phi(g_2)} \longrightarrow P_{g_1g_2}$$

together with the condition that $\phi : G \longrightarrow \mathbb{Z}_2$ is a group homomorphism. Finally, there is a diagram over $G \times G \times G$. If ϕ is trivial, then (P,λ) is a multiplicative S^1-bundle over G, i.e. a central extension. With non-trivial ϕ, it is what Freed and Moore call a ϕ-twisted extension of G.

If the same discussion is repeated with $\mathrm{AUT}(S^1) \times \mathbb{Z}_2$, we also get a ϕ-twisted extension of G together with an independent map $c : G \longrightarrow \mathbb{Z}_2$. Now let us consider a twisted vector bundle for a ϕ-twisted extension of G and the map c. By definition it is a 1-morphism $\mathcal{E} : \mathcal{I} \longrightarrow \mathcal{I}$ in $\mathcal{G}rb_{gr\mathbb{C},\mathbb{Z}_2}(pt)$, i.e. a $\mathbb{Z}_2$-graded hermitian vector space E, together with a sign $e \in \mathbb{Z}_2$, and a 2-isomorphism $\rho : \mathsf{pr}^*\mathcal{E} \circ \mathcal{A} \Longrightarrow \mathsf{pr}^*\mathcal{E}$, where $\mathsf{pr} : G \longrightarrow pt$. Spelling this our, it is for each $g \in G$ a graded linear isometry

$$\rho_g : (E \otimes P_g^e)^{\phi(g)} \longrightarrow E$$

where P_g is graded according to $c(g)$. As seen above, this can be reformulated by saying that ρ_g is even or odd depending on $c(g)$. With $e = 1$, this structure is precisely the one of a (ϕ, c)-twisted central extension as used by Freed and Moore.

Literature and further reading: [FM]

Remark 3.2.5. A multiplicative smooth functor $F : \Gamma \longrightarrow \Omega$ induces a 2-functor

$$\mathcal{G}rb_\Gamma \longrightarrow \mathcal{G}rb_\Omega.$$

If Γ is a Lie 2-group, we always have the two functors

$$\Gamma \longrightarrow \pi_0\Gamma \quad \text{and} \quad B\pi_1\Gamma \longrightarrow \Gamma.$$

If G is a Lie group, regarded as a Lie 2-group with only identity morphisms, then

$$\mathcal{G}rb_G = \mathcal{B}un_G.$$

(Notice the clash of notation: $\mathcal{G}rb_{S^1}$ is of course not $\mathcal{B}un_{S^1}$: there, S^1 was regarded as BS^1 whereas we regard it here as the groupoid with only identity morphisms.) The $\mathbb{Z}_2$-bundles which appeared above are obtained this way:

$$\mathcal{G}rb_\Gamma(M) \longrightarrow \mathcal{G}rb_{\mathbb{Z}_2}(M) = \mathcal{B}un_{\mathbb{Z}_2}(M).$$

3.3 Connections I: Trivial gerbes

As before, we have obtained the sheaf of non-abelian gerbes by sheafification of the presheaf

$$\mathcal{T}riv\mathcal{G}rb_\Gamma := B\mathcal{B}un_\Gamma$$

of trivial Γ-gerbes. In order to say what connections on non-abelian gerbes are, we only need to know the presheaf of trivial Γ-gerbes with connections.

In case of abelian gerbes, we have motivated the definition of a connection on the trivial gerbe (a 2-form $B \in \Omega^2(M)$) by saying that we wanted to define a surface holonomy. In the non-abelian case, this is slightly more complicated since one would not expect something like a Γ-valued holonomy function.

In order to illustrate that let us consider the transition from S^1-bundles to G-bundles, for G a non-abelian group. What would we expect from a connection on a G-bundle, or rather, a trivial principal G-bundle? We expect that it has a well-defined, G-valued parallel transport, a map

$$PM \longrightarrow G$$

assigning to each path $\gamma \in PM$ in the manifold an element of G. It turns out that the usual properties of parallel transport can be encoded by saying that this map makes up a smooth functor

$$F : \mathcal{P}_1(M) \longrightarrow BG$$

defined on the path groupoid $\mathcal{P}_1(M)$ of M with objects the points of M and morphisms the paths, to the groupoid BG.

Literature and further reading: [CP94, SW09]

Via a process of differentiation, one can translate from the language of smooth functors into the language of differential forms:

Theorem 3.3.1 ([SW09])**.** *There is an isomorphism of categories:*

$$\mathcal{F}un^\infty(\mathcal{P}_1(M), BG) = \mathcal{C}onn_G(M),$$

where $\mathcal{C}onn_G(M)$ is the category of G-connections on M, i.e. the objects are 1-forms $A \in \Omega^1(M, \mathfrak{g})$ with values in the Lie algebra of G, and the morphisms $A \longrightarrow A'$ are smooth functions $g : M \longrightarrow G$ such that $A' = \mathrm{Ad}_g(A) - g^*\bar{\theta}$.

Here and in the following, $g^*\theta = g^{-1}dg$ and $g^*\bar{\theta} = dgg^{-1}$ stand for the pullbacks of the left- and right-invariant Maurer-Cartan forms on G. Following our philosophy, we would now define

$$\mathcal{T}riv\mathcal{B}un_G^\nabla := \mathcal{C}onn_G \quad \text{and} \quad \mathcal{B}un_G^\nabla := (\mathcal{T}riv\mathcal{B}un_G^\nabla)^+$$

Exercise: Examine in detail the sheaf $(\mathcal{T}riv\mathcal{B}un_G^\nabla)^+$ by performing the plus construction with the presheaf of G-connections. Show that it is equivalent to the sheaf of G-bundles with connection as usually defined.

After this motivation, let us return to the problem of defining connections on non-abelian gerbes. One can consider a bigroupoid $\mathcal{P}_2(M)$, the path 2-groupoid of M, whose objects are the points of M, whose morphisms are the paths in M, and whose 2-morphisms are (fixed ends) homotopies between paths. If Γ is a Lie 2-group, let $B\Gamma$ denote the associated bigroupoid with one object. Now we look for smooth 2-functors

$$F : \mathcal{P}_2(M) \longrightarrow B\Gamma.$$

Such 2-functors model trivial Γ-gerbes with connection.

Suppose $\Gamma = G/\!/H$ comes from a crossed module $t : H \longrightarrow G$. If $\gamma \in PM$ is a path, we understand $F(\gamma) \in G$ as the parallel transport of the trivial gerbe along γ. If $\Sigma : \gamma \Longrightarrow \gamma'$ is a homotopy between paths, we obtain elements $(h, g) := F(\Sigma)$, with $g = F(\gamma)$ and $t(h)g = F(\gamma')$. Then we understand the element $h \in H$ as the parallel transport of the trivial gerbe along the "surface" Σ. The axioms of the 2-functor F assure appropriate gluing properties of these parallel transport assignments.

One can now translate the bigroupoid $\mathcal{F}un^\infty(\mathcal{P}_2(M), B\Gamma)$ into differential forms and smooth functions. The result is the following:

Theorem 3.3.2 ([SW11])**.** *Let $\Gamma = G/\!/H$ be a Lie 2-group coming from a crossed module $t : H \longrightarrow G$, where $\alpha : G \times H \longrightarrow H$ is the G-action on H. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Then, the bigroupoid $\mathcal{F}un^\infty(\mathcal{P}_2(M), B\Gamma)$ is isomorphic to a bigroupoid defined as follows:*

1. *An object is a pair (A, B) of a 1-form $A \in \Omega^1(X, \mathfrak{g})$ and a 2-form $B \in \Omega^2(X, \mathfrak{h})$ such that*
$$\mathrm{d}A + [A \wedge A] = t_* \circ B.$$
2. *If (A, B) and (A', B') are two objects, the category*
$$\mathcal{H}om((A, B), (A', B'))$$
is defined as follows:
   (a) *An object is a pair (g, φ) of a smooth map $g : X \longrightarrow G$ and a 1-form*

$\varphi \in \Omega^1(X, \mathfrak{h})$ *such that*

$$\begin{aligned} A' + t_* \circ \varphi &= \mathrm{Ad}_g(A) - g^*\bar{\theta} \\ B' + \alpha_*(A' \wedge \varphi) + \mathrm{d}\varphi + [\varphi \wedge \varphi] &= (\alpha_g)_* \circ B. \end{aligned}$$

The identity is given by $g = 1$ *and* $\varphi = 0$.

(b) A morphism $a : (g_1, \varphi_1) \Longrightarrow (g_2, \varphi_2)$ *is a smooth map* $a : X \longrightarrow H$ *such that:*

$$g_2 = (t \circ a) \cdot g_1 \quad \text{and} \quad \varphi_2 + (r_a^{-1} \circ \alpha_a)_*(A') = \mathrm{Ad}_a(\varphi_1) - a^*\bar{\theta}.$$

The composition

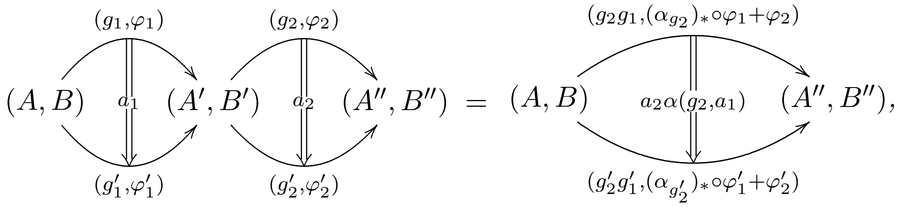
$$(g, \varphi) \xrightarrow{a_1} (g', \varphi') \xrightarrow{a_2} (g'', \varphi'')$$

is given by $a_2 a_1$.

3. *The composition functor*

$$\mathcal{H}om((A', B'), (A'', B'')) \times \mathcal{H}om((A, B), (A', B')) \longrightarrow \mathcal{H}om((A, B), (A'', B''))$$

is defined on objects by $g_2 g_1 : X \longrightarrow G$ *and the 1-form* $(\alpha_{g_2})_* \circ \varphi_1 + \varphi_2$. *On morphisms it is given by*



and the identity 2-morphism is given by $a = 1$.

Remark 3.3.3. The condition

$$\mathrm{d}A + [A \wedge A] = t_* \circ B$$

is called the fake flatness of the connection (A, B). Some authors prefer not to include it into the definition. Indeed, since this equation is quadratic, it implies that the space of connections is not affine. In particular, not every non-abelian gerbe has a connection, and if it does, the space of possible connections will not be contractible.

Literature and further reading: [MP02, BM05, SW11, ACJ05]

3.4 Connections II: Sheafification of the Hom-categories

Now we begin to sheafify the presheaf $\mathcal{F}un^\infty(\mathcal{P}_2(M), B\Gamma)$. We'll be explicit, and start by sheafifying the Hom-categories first, performing the plus construction. Thus, an object in $\mathcal{H}om((A,B),(A',B'))^+$ is a surjective submersion $\pi : Y \longrightarrow X$, a 1-form $\varphi \in \Omega^1(Y,\mathfrak{h})$, a smooth map $g : Y \longrightarrow G$, and a smooth map $a : Y^{[2]} \longrightarrow H$ satisfying the condition for morphisms,

$$g(y_2) = t(a(y_1,y_2)) \cdot g(y_1) \quad , \quad \mathsf{pr}_2^*\varphi + (r_a^{-1} \circ \alpha_a)_*(\mathsf{pr}_M A') = \mathrm{Ad}_a(\mathsf{pr}_1^*\varphi) - a^*\bar{\theta},$$

and the coherence over $Y^{[3]}$:

$$a(y_2,y_3) \cdot a(y_1,y_2) = a(y_3,y_1).$$

As usually, one translates this into a principal H-bundle P with total space

$$P := (Y \times H)/\sim \qquad \text{where} \qquad (y_1,h) \sim (y_2, a(y_1,y_2)h),$$

and right action $(y,h) \cdot h' := (y, hh')$.

Exercise: Show that the map

$$\phi : Y \times H \longrightarrow G : (y,h) \longmapsto t(h)^{-1} \cdot g(y)$$

descends to an H-anti-equivariant map $\phi : P \longrightarrow G$.

By the exercise, we see that (P,ϕ) is a $G/\!\!/H$-bundle over M. Next uncover what the 1-form $\varphi \in \Omega^1(Y,\mathfrak{h})$ gives. For this purpose we use the remaining identity

$$\mathsf{pr}_2^*\varphi + (r_a^{-1} \circ \alpha_a)_*(\mathsf{pr}_M A') = \mathrm{Ad}_a(\mathsf{pr}_1^*\varphi) - a^*\bar{\theta}$$

Exercise: Let $\mathsf{pr}_Y : Y \times H \longrightarrow Y$ denote the projection, let $\mathsf{pr}_M := \pi \circ \mathsf{pr}_Y$, and let $h : Y \times H \longrightarrow H$ denote the projection to the second factor. Show that the 1-form

$$\omega := \mathrm{Ad}_h^{-1}(\mathsf{pr}_Y^*\varphi + (r_h^{-1} \circ \alpha_h)_*(\mathsf{pr}_M^* A')) + h^*\theta \in \Omega^1(Y \times H, \mathfrak{h})$$

descends to P and satisfies the following axioms:

(i) We have the three maps $\mathsf{pr}_P : P \times H \longrightarrow P$ (projection), $\rho : P \times H \longrightarrow P$ (the principal H-action), and $h : P \times H \longrightarrow H$ (the other projection), and ω satisfies:

$$\rho^*\omega = \mathrm{Ad}_h^{-1}(\mathsf{pr}_P^*\omega) + (l_{h'}^{-1} \circ \alpha_{h'})_*(\mathsf{pr}_M^* A') + h^*\theta.$$

Note that this is an "A'-twisted version" of an ordinary connection on P, whose defining equation is

$$\rho^*\omega = \mathrm{Ad}_h^{-1}(\mathsf{pr}_P^*\omega) + h^*\theta.$$

(ii) It satisfies the following equation for $\mathfrak{g}$-valued forms on P:

$$t_*(\omega) = \mathrm{Ad}_\phi(\mathsf{pr}_M^* A) - \mathsf{pr}_M^* A' - \phi^*\bar{\theta}.$$

(iii) Its curvature is

$$d\omega + [\omega \wedge \omega] = \alpha_*(\mathsf{pr}_M^* A' \wedge \omega) + B' + (\alpha_\phi)_*(B).$$

Note: These exercises are a lot of tedious work!

Summarizing, the objects of $\mathcal{H}om((A,B),(A',B'))^+$ are H-bundles P with H-anti-equivariant smooth maps $\phi : P \longrightarrow G$, and an A'-twisted connection ω satisfying (ii) and (iii) of above list. Let us next look at the morphisms.

First we ignore the forms. We consider two objects (Y_1, π_1, g_1, a_1) and (Y_2, π_2, g_2, a_2). A morphism is a common refinement $\zeta : Z \longrightarrow M$ of $\pi_1 : Y_1 \longrightarrow M$ and $\pi_2 : Y_2 \longrightarrow M$, together with a smooth map $h : Z \longrightarrow H$ such that

$$g_2(y_2(z)) = t(h(z)) \cdot g_1(y_1(z))$$

and

$$a_2(y_2(z), y_2(z')) \cdot h(z) = h(z') \cdot a_1(y_1(z), y_1(z')).$$

We want to define an isomorphism $f : P_1 \longrightarrow P_2$ of the corresponding H-bundles P_1 and P_2. For $(y_1, h) \in P_1$ choose $z \in Z$ with $y_1(z) = y_1$ and set

$$f(y_1, h) := (y_2(z), h(z) \cdot h).$$

Exercise: Show that this defines a smooth map $f : P_1 \longrightarrow P_2$ that preserves the bundle projections, the H-actions, and the anchors; that is: a morphism between $G/\!\!/H$-bundles over M.

Exercise: Show that f respects the A'-twisted connections, i.e. $f^*\omega_2 = \omega_1$.

Example 3.4.1. Consider $\Gamma = BS^1$, so that $G = 1$ and $H = S^1$. Reducing the data of the last theorem to this case, the bicategory $\mathcal{T}riv\mathcal{G}rb_\Gamma^\nabla(M)$ has as objects 2-forms $B \in \Omega^2(M)$, as 1-morphisms $B \longrightarrow B'$ 1-forms $\varphi \in \Omega^1(M)$ such that $d\varphi = B - B'$, and as 2-morphisms $\varphi_1 \Longrightarrow \varphi_2$ smooth maps $a : M \longrightarrow S^1$ such that $\varphi_2 = \mathrm{Ad}_a(\varphi_1) - a^*\bar{\theta}$. The first step in the sheafification procedure, the sheafification of the Hom-presheaves that we have performed above, leads precisely to the bicategory $\mathcal{T}riv\mathcal{G}rb^{\nabla}_{S^1}$ considered in Lecture II.

Literature and further reading: [BM05, ACJ05, SW11, SSW07]

3.5 Connections III: Full sheafification

We have now obtained, by sheafification of the Hom-categories of the presheaf $\mathcal{F}un^\infty(\mathcal{P}_2(M), B\Gamma)$ the presheaf $\mathcal{T}riv\mathcal{G}rb^{\nabla}_{\Gamma}$ of trivial Γ-gerbes with connection. We assume again that $\Gamma = G/\!\!/H$ is a crossed module. We recall that this presheaf assigns to a smooth manifold a bicategory with:

(i) Objects: pairs (A, B) satisfying the fake-flatness condition.

(ii) 1-morphisms from (A, B) to (A', B'): H-bundles P with H-anti-equivariant smooth maps $\phi : P \longrightarrow G$ and with A'-twisted connections ω satisfying conditions (ii) and (iii) from above list.

(iii) 2-morphisms: the connection-preserving, anchor-preserving smooth bundle morphisms.

Notice that, upon forgetting all differential form data, this is $B\mathcal{B}un_{\Gamma}(M)$. Now we define

$$\mathcal{G}rb^{\nabla}_{\Gamma} := (\mathcal{T}riv\mathcal{G}rb^{\nabla}_{\Gamma})^+ .$$

So, a Γ-gerbe with connection consists of:

(i) a surjective submersion $\pi : Y \longrightarrow M$.

(ii) over Y, differential forms $A \in \Omega^1(Y, \mathfrak{g})$ and $B \in \Omega^2(Y, \mathfrak{h})$ such that

$$\mathrm{d}A + [A \wedge A] = t_* \circ B.$$

(iii) over $Y^{[2]}$, a principal H-bundle P with anchor $\phi : P \longrightarrow G$ and $\mathsf{pr}_2^* A$-twisted connection ω, such that

$$\begin{aligned} t_*(\omega) &= \mathrm{Ad}_\phi(\mathsf{pr}_1^* A) - \mathsf{pr}_2^* A - \phi^*\bar{\theta}. \\ d\omega + [\omega \wedge \omega] &= \alpha_*(\mathsf{pr}_2^* A \wedge \omega) + \mathsf{pr}_2^* B + (\alpha_\phi)_*(\mathsf{pr}_1^* B). \end{aligned}$$

(iv) over $Y^{[4]}$, a bundle morphism

$$\mu : \mathsf{pr}_{23}^*(P,\phi) \otimes \mathsf{pr}_{12}^*(P,\phi) \longrightarrow \mathsf{pr}_{13}^*(P,\phi).$$

that preserves anchors and connections, and satisfies an associativity constraint over $Y^{[4]}$.

Literature and further reading: [BM05, SW, ACJ05]

Remark 3.5.1. Recall that if Γ is a smoothly separable Lie 2-group then its zeroth homotopy group $\pi_0\Gamma$ is a Lie group and the projection $p : \Gamma_0 \longrightarrow \pi_0\Gamma$ is a surjective submersion. If we represent Γ by a crossed modules, $\Gamma = G/\!\!/H$, then $\pi_0\Gamma = G/H$. As explained earlier, if $\mathcal{G}$ is a Γ-gerbe over M, we have a $\pi_0\Gamma$-bundle $\pi_0(\mathcal{G})$ over M. It is produced from $\pi : Y \longrightarrow M$ and the cocycle

$$\tilde{\phi} : Y^{[2]} \longrightarrow G/H$$

which is obtained by letting $p \circ \phi : P \longrightarrow G/H$ descend. Now suppose $\mathcal{G}$ carries a connection, consisting, in particular, of a 1-form $A \in \Omega^1(Y, \mathfrak{g})$ satisfying

$$t_*(\omega) = \mathrm{Ad}_\phi(\mathsf{pr}_1^*A) - \mathsf{pr}_2^*A - \phi^*\bar{\theta},$$

for ω the pr_2^*A-twisted connection on P. Composing this equation with $p_* : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ yields

$$\mathsf{pr}_2^*A = \mathrm{Ad}_\alpha(\mathsf{pr}_1^*A) - \bar{\theta}_\alpha,$$

i.e., A defines a connection on $\pi_0(\mathcal{G})$. The fake-flatness condition

$$\mathrm{d}A + [A \wedge A] = t_* \circ B$$

implies under p_* that this connection on $\pi_0(\mathcal{G})$ is flat.

Example 3.5.2. Earlier we have seen what an $\mathrm{AUT}(S^1)$-gerbe is; it consisted of a $\mathbb{Z}_2$-graded S^1-bundle (P,ϕ) over $Y^{[2]}$ and a bundle isomorphism

$$\mu : \mathsf{pr}_{23}^*P \otimes \mathsf{pr}_{12}^*P^{\phi\circ\mathsf{pr}_{23}} \longrightarrow \mathsf{pr}_{13}^*P.$$

Now let us discuss connection data. Since the Lie algebra $\mathfrak{g}$ of $G = \mathbb{Z}_2$ is trivial, we just have the 2-form $B \in \Omega^2(Y)$. Thus, the bundle P over $Y^{[2]}$ carries an ordinary (untwisted) connection ω. The relation between the curvature of ω and the 2-form B becomes via the calculation $(\alpha_\phi)_*(X) = \phi X$

$$\mathrm{curv}(\omega) = \mathsf{pr}_2^* B + \phi \cdot (\mathsf{pr}_1^* B).$$

The bundle isomorphism μ is required to be connection-preserving.

Example 3.5.3. Now we can consider twistings of a differential version of Atiyah's KR-theory. In physicists terminology, these are background fields for bosonic orientifolds. We are concerned with a smooth manifold M and an involution $k : M \longrightarrow M$, which we regard as a $\mathbb{Z}_2$-action on M, so that we have the associated action groupoid $M/\!\!/k$.

(a) We notice that the trivial $\mathbb{Z}_2$-bundle $M \times \mathbb{Z}_2$ over M can be equipped with the $\mathbb{Z}_2$-equivariant structure which changes the sign under the involution. So it is a $\mathbb{Z}_2$-bundle over $M/\!\!/k$, and we denote it by $Or(M/\!\!/k)$, the orientation bundle of $M/\!\!/k$.

(b) A *Jandl gerbe* is an $\mathrm{AUT}(S^1)$-gerbe $\mathcal{G}$ with connection over $M/\!\!/k$ and a bundle isomorphism

$$\pi_0(\mathcal{G}) \cong Or(M/\!\!/k).$$

Jandl gerbes are the B-fields for bosonic orientifolds, or twistings of differential KR-theory.

(c) Let us spell out in detail the data of a Jandl gerbe. We shall first see that the requirement that the $\mathrm{AUT}(S^1)$-gerbe over M has trivial a trivializable $\mathbb{Z}_2$-bundle $\pi_0(\mathcal{G})$ implies that it is reducible to an ordinary S^1-gerbe.

The data of the underlying $\mathrm{AUT}(S^1)$-gerbe has been discussed earlier; it involved, in particular, anchor map $\phi : Y^{[2]} \longrightarrow \mathbb{Z}_2$. A choice of a bundle morphism $\pi_0(\mathcal{G}) \cong Or(M/\!\!/k)$ determines a map $\psi : Y \longrightarrow \mathbb{Z}_2$ with $\phi(y_1, y_2) = \psi(y_1)\psi(y_2)$. Let P_ψ denote the trivial S^1-bundle over Y, equipped with the anchor map ψ. Then we pass to the $\mathrm{AUT}(S^1)$-bundle

$$P_{\mathrm{red}} := \mathsf{pr}_2^* P_\psi \otimes P \otimes \mathsf{pr}_1^* P_\psi.$$

Exercise: Check, using the rules for tensor products of groupoid bundles, that this gives a new $\mathrm{AUT}(S^1)$-bundle which has a trivial anchor and so is an ordinary S^1-bundle. Verify that as manifolds $P_{\mathrm{red}} = P$, but with the S^1-action given by $p \star z := p \cdot \alpha(\psi \circ \mathsf{pr}_2, z)$.

Exercise: Check that $\mu_{\text{red}} := \mu^{\psi \circ \mathsf{pr}_3}$ gives a bundle gerbe product for P_{red}, i.e.

$$\mu_{\text{red}} : \mathsf{pr}_{23}^* P_{\text{red}} \otimes \mathsf{pr}_{12}^* P_{\text{red}} \longrightarrow \mathsf{pr}_{13}^* P_{\text{red}}.$$

Thus, $\mathcal{G}_{\text{red}} := (Y, P_{\text{red}}, \mu_{\text{red}})$ is an ordinary S^1-bundle gerbe. We equip the trivial bundle P_ψ with the trivial connection, so that P_{red} carries the tensor product connection, and μ_{red} is connection-preserving.

Exercise: Show that $\text{curv}(P_{\text{red}}) = (\psi \circ \mathsf{pr}_2) \cdot \text{curv}(P)$. For $B \in \Omega^2(Y)$ the curving of $\mathcal{G}$, define $B_{\text{red}} := \psi \cdot B$ and show that

$$\text{curv}(P_{\text{red}}) = \mathsf{pr}_1^* B_{\text{red}} - \mathsf{pr}_2^* B_{\text{red}}.$$

Summarizing, $\mathcal{G}_{\text{red}}$ is an ordinary S^1-gerbe with an ordinary connection. We continue using $\mathcal{G}_{\text{red}}$ instead of $\mathcal{G}$ (this can be justified because $\mathcal{G}$ and $\mathcal{G}_{\text{red}}$ are isomorphic as $\text{AUT}(S^1)$-gerbes).

(d) Next we take care about the $\mathbb{Z}_2$-equivariant structure. We have an isomorphism

$$\mathcal{A} : s^* \mathcal{G}_{\text{red}} \longrightarrow t^* \mathcal{G}_{\text{red}}$$

over the morphism space $\mathbb{Z}_2 \times M$ of the action groupoid $M /\!/ k$. We may consider that space as $M \coprod M$ with $s = t = \text{id}_M$ on the first copy and $s = \text{id}_M$ and $t = k$ on the second copy. Accordingly, $\mathcal{A}$ has two components $\mathcal{A}_{\text{id}} : \mathcal{G}_{\text{red}} \longrightarrow \mathcal{G}_{\text{red}}$ and $\mathcal{A}_k : \mathcal{G}_{\text{red}} \longrightarrow k^* \mathcal{G}_{\text{red}}$. These are 1-morphisms in $\mathcal{G}rb^{\nabla}_{\text{AUT}(S^1)}(M)$, although $\mathcal{G}_{\text{red}}$ is an S^1-gerbe. So, they come with refinements $Z_{\text{id}} \longrightarrow M$ and $Z_k \longrightarrow M$ and $\text{AUT}(S^1)$-bundles $(Q_{\text{id}}, \phi_{\text{id}})$ and (Q_k, ϕ_k), respectively. We recall that our choice of a bundle morphism $\pi_0(\mathcal{G}) \cong Or(M /\!/ k)$ determined a map $\psi : Y \longrightarrow \mathbb{Z}_2$. That this bundle morphism is supposed to be k-equivariant implies that $\phi_{\text{id}} = 1$ and $\phi_k = -1$.

Let us focus first on $\mathcal{A}_{\text{id}}$. Since $\phi_{\text{id}} = 1$, $\mathcal{A}_{\text{id}}$ is actually a 1-morphism in $\mathcal{G}rb^{\nabla}_{S^1}(M)$. For the equivariant structure there is a coherence 2-isomorphism

$$\varphi : \mathsf{pr}_2^* \mathcal{A} \circ \mathsf{pr}_1^* \mathcal{A} \longrightarrow c^* \mathcal{A}$$

over the manifold of composable morphisms of $M /\!/ k$. It can be restricted to a 2-isomorphism $\mathcal{A}_{\text{id}} \circ \mathcal{A}_{\text{id}} \cong \mathcal{A}_{\text{id}}$, showing that $\mathcal{A}_{\text{id}} = \text{id}_{\mathcal{G}_{\text{red}}}$. Thus $\mathcal{A}_{\text{id}}$ is actually no information.

The interesting part of the $\mathbb{Z}_2$-equivariant structure is $\mathcal{A}_k : \mathcal{G}_{\mathrm{red}} \longrightarrow k^*\mathcal{G}_{\mathrm{red}}$. We may assume that the surjective submersion of $k^*\mathcal{G}_{\mathrm{red}}$ is $\pi_k := k \circ \pi : Y \longrightarrow M$, and that the refinement Z_k is just $Y\, {}_{\pi}\!\times_{\pi_k} Y$ with the projections as refinement maps. Spelling out what the graded tensor products for $\phi_k = -1$ means, we see that $\mathcal{A}_k$ consists of the S^1-bundle Q_k over Z_k and of a connection-preserving bundle isomorphism

$$\alpha : \mathsf{pr}_{24}^* Q_k \otimes \mathsf{pr}_{12}^* P_{\mathrm{red}}^{\vee} \longrightarrow \mathsf{pr}_{34}^* P_{\mathrm{red}} \otimes \mathsf{pr}_{13}^* Q_k$$

over $Z_k^{[2]} \cong Y^{[2]} \times_M Y_k^{[2]}$. The condition for the curvings is

$$\mathrm{curv}(Q) = \mathsf{pr}_1^* B_{\mathrm{red}} + \mathsf{pr}_2^* B_{\mathrm{red}}.$$

This is the same as a 1-morphism

$$\mathcal{A}_k : \mathcal{G}_{\mathrm{red}}^{\vee} \longrightarrow k^*\mathcal{G}_{\mathrm{red}}$$

in $\mathcal{G}rb_{S^1}^{\nabla}(M)$, where $\mathcal{G}^{\vee}$ denotes the dual gerbe. Finally, we restrict the 2-isomorphism φ to a 2-isomorphism

$$\varphi_k : k^*\mathcal{A}_k \circ \mathcal{A}_k^{\vee} \Longrightarrow \mathrm{id}_{\mathcal{G}_{\mathrm{red}}},$$

and the final coherence condition is $k^*\varphi_k^{-1} = \varphi_k^{\vee}$.

(e) Summarizing, we have seen that a Jandl gerbe $\mathcal{G}$ over $M/\!\!/k$, i.e. an $\mathrm{AUT}(S^1)$-gerbe with connection over $M/\!\!/k$ such that $\pi_0(\mathcal{G}) \cong Or(M/\!\!/k)$, is the same as:

(a) An S^1-gerbe with connection over M.

(b) A 1-isomorphism $\mathcal{A} : \mathcal{G}^{\vee} \longrightarrow k^*\mathcal{G}$.

(c) A 2-isomorphism $\varphi : k^*\mathcal{A} \circ \mathcal{A}^{\vee} \Longrightarrow \mathrm{id}_{\mathcal{G}}$ such that $k^*\varphi^{-1} = \varphi^{\vee}$.

This is precisely the data of a "bundle gerbe with a Jandl structure", for which a notion of surface holonomy can be defined for unoriented surfaces. More precisely, if Σ is a possibly unoriented surface, it assigns a well-defined element in S^1 to each differentiable stack map $\phi : \Sigma \longrightarrow M/\!\!/k$, i.e. to a $\mathbb{Z}_2$-bundle $\hat{\Sigma}$ over Σ and a $\mathbb{Z}_2$-equivariant map $\hat{\phi} : \hat{\Sigma} \longrightarrow M$. This surface holonomy constitutes the contribution of the orientifold B-field to the sigma model action.

Literature and further reading: [SSW07, Wal07, FNSW08, NWb, NS11, GSW11]

3.6 2-Groupoid gerbes and a two-fold delooping of the circle

In this last part we would like to incorporate a further twist of K-theory by a $\mathbb{Z}_2$-factor in degree zero, the so-called grading twist. In order to do so, we have to go one further step down in the table of structure things presented at the beginning of Lecture III: we have to admit gerbes with 2-groupoidal structure.

Luckily, we have all the machinery at our disposal. If $\mathscr{G}$ is an arbitrary Lie 2-groupoid, we have to look at the bicategory

$$Fun^{\infty}(\mathcal{P}_2(X), \mathscr{G})$$

of smooth 2-functors, considered as the bicategory of trivial $\mathscr{G}$-gerbes with connection. Right now there is no written account for the translation of this bicategory into smooth functions and differential forms, generalizing Theorem 3.3.2.

We continue this discussion with the case that $\mathscr{G}$ arises by composition of two abstract constructions: it is the horizontal bicategory of an action double groupoid. Just like the action groupoid, the action double groupoid is obtained from an action of a Lie 2-group Γ on a smooth category $\mathcal{C}$. In the simplest case, the action is a smooth functor

$$R : \Gamma \times \mathcal{C} \longrightarrow \mathcal{C},$$

that satisfies strictly the axioms of an action. The action double groupoid $\mathcal{C}/\!\!/\Gamma$ is defined as follows:

(i) Its category of objects is $\mathcal{C}$.

(ii) Its category of morphisms is $\Gamma \times \mathcal{C}$, source is the projection pr_2, and target is the action R.

(iii) The composition is the multiplication in Γ.

The horizontal bicategory $\mathscr{G} := \mathscr{H}(\mathcal{C}/\!\!/\Gamma)$ looks as follows:

(i) The objects: $\mathscr{G}_0 := \mathcal{C}_0$.

(ii) 1-morphisms are: $\mathscr{G}_1 := \Gamma_0 \times \mathcal{C}_0$, where a pair $(g, x) \in \Gamma_0 \times \mathcal{C}_0$ is considered as a morphism from $s(g, x) := x$ to $t(g, x) := R_0(g, x)$. The composition is the multiplication in Γ_0.

(iii) 2-morphisms are

$$\mathscr{G}_2 := \{(\gamma, x) \in \Gamma_1 \times \mathcal{C}_0 \mid R_0(s(\gamma), x) = R_0(t(\gamma), x)\}$$

Such a pair (γ, x) is considered as a morphism from $s(\gamma, x) := (s(\gamma), x)$ to $t(\gamma, x) := (t(\gamma), x)$. The vertical composition is the composition in Γ, and the horizontal composition is the multiplication of Γ_1.

Example 3.6.1. We consider two actions

$$R : \mathbb{Z}_2 \times \mathrm{AUT}(S^1) \longrightarrow \mathrm{AUT}(S^1).$$

1. The first action is defined on objects by $R_0(x, y) := xy$ and on morphisms by $R_1(\mathrm{id}_x, (z, y)) := (z, xy)$. Thus, the associated Lie 2-groupoid $\mathscr{G}$ has as objects $\mathscr{G}_0 = \mathbb{Z}_2$ and as 1-morphisms $\mathscr{G}_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ with source, target, and composition given by $s(x, y) = y$, $t(x, y) = xy$ and $(z, xy) \circ (x, y) = (zx, y)$. It has as 2-morphisms $\mathscr{G}_2 = S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with source and target both given by $(z, x, y) \longmapsto (x, y)$. Vertical composition is
$$(z_2, x, y) \bullet (z_1, x, y) = (z_2 z_1, x, y),$$
and the horizontal composition is
$$(z_2, x_2, y_2) \circ (z_1, x_1, y_1) = (z_2 z_1^{x_2}, x_2 x_1, y_1).$$
This Lie 2-groupoid $\mathscr{G}$ has the homotopy groups
$$\pi_0 \mathscr{G} = 1 \quad , \quad \pi_1 \mathscr{G} = 1 \quad \text{and} \quad \pi_2 \mathscr{G} = S^1.$$

2. The second action is the trivial one, $R_{0(x,y)} = y$. The manifolds $\mathscr{G}_0$, $\mathscr{G}_1$ and $\mathscr{G}_2$ are as before, but a 1-morphism (x, y) has source and target y, and the composition is given by $(z, y) \circ (x, y) = (zx, y)$. In this case, the Lie 2-groupoid $\mathscr{G}$ has the homotopy groups
$$\pi_0 \mathscr{G} = \mathbb{Z}_2 \quad , \quad \pi_1 \mathscr{G} = \mathbb{Z}_2 \quad \text{and} \quad \pi_2 \mathscr{G} = S^1.$$

Now let us look at the bicategory $Fun^\infty(\mathcal{P}_2(X), \mathscr{G})$, for $\mathscr{G}$ obtained from an action of a Lie 2-group $\Gamma = G/\!\!/H$ on a smooth category $\mathcal{C}$. Generalizing the procedure developed in [SW11] that we have used earlier in the case $\mathscr{G} = B\Gamma$, we get the following translation into differential forms and smooth functions:

(i) An object is a triple (f, A, B) consisting of a smooth map $f : X \longrightarrow \mathcal{C}_0$, and of differential forms $A \in \Omega^1(X, \mathfrak{g})$ and $B \in \Omega^2(X, \mathfrak{h})$ satisfying the fake-flatness-condition. They have to be compatible with the action R in the sense that

$$\rho_A \circ f = df \quad \text{and} \quad \rho_{t_* B} \circ f = 0,$$

where

$$\rho_A : \mathcal{C}_0 \longrightarrow T\mathcal{C}_0 : m \longmapsto \left.\frac{d}{dt}\right|_0 R_0(e^{-At}, m)$$

is the infinitesimal action of $\mathfrak{g}$ on $\mathcal{C}_0$.

(ii) A 1-morphism $(f, A, B) \longrightarrow (f', A', B')$ is a smooth map $g : X \longrightarrow G$ such that $f'(x) = R_0(g(x), f(x))$ and a 1-form $\varphi \in \Omega^1(X, \mathfrak{h})$ satisfying the same equations as in the case $\mathscr{G} = B\Gamma$,

$$\begin{aligned} A' + t_* \circ \varphi &= \mathrm{Ad}_g(A) - g^*\bar{\theta} \\ B' + \alpha_*(A' \wedge \varphi) + \mathrm{d}\varphi + [\varphi \wedge \varphi] &= (\alpha_g)_* \circ B, \end{aligned}$$

and additionally

$$\rho_{t_*\varphi} \circ f = 0.$$

(iii) Let $(g_1, \varphi_1), (g_2, \varphi_2) : (f, A, B) \longrightarrow (f', A', B')$ be 1-morphisms. 2-morphisms exist if $R_0(g_1(x), f(x)) = R_0(g_2(x), f(x))$. In that case, a 2-morphism $(g_1, \varphi_1) \Longrightarrow (g_2, \varphi_2)$ is a smooth map $a : X \longrightarrow H$ with satisfying the conditions of the case $\mathscr{G} = B\Gamma$:

$$g_2 = (t \circ a) \cdot g_1 \quad \text{and} \quad \varphi_2 + (r_a^{-1} \circ \alpha_a)_*(A') = \mathrm{Ad}_a(\varphi_1) - a^*\bar{\theta}.$$

Finally, we shall look at a $\mathscr{G}$-gerbe with connection, with $\mathscr{G}$ the Lie 2-groupoids obtained from the two action of $\mathbb{Z}_2$ on $\mathrm{AUT}(S^1)$ discussed above. In case of the first (non-trivial) action, a $\mathscr{G}$-gerbe with connection is:

(a) A surjective submersion $\pi : Y \longrightarrow M$.

(b) A smooth map $f : Y \longrightarrow \mathbb{Z}_2$ and a 2-form $B \in \Omega^2(Y)$.

(c) Over $Y^{[2]}$, a smooth map $\phi : Y^{[2]} \longrightarrow \mathbb{Z}_2$ such that $f(y_2) = \phi(y_1, y_2) f(y_1)$ and an S^1-bundle P with connection ω of curvature

$$\mathrm{curv}(\omega) = \mathsf{pr}_2^* B + \phi \cdot (\mathsf{pr}_1^* B).$$

(d) Over $Y^{[3]}$, a connection-preserving bundle gerbe product.

The situation here is similar to the example with the Jandl gerbe: one can pass to an ordinary S^1-gerbe, reflecting the fact that $\mathscr{G}$ is (weakly) equivalent to BBS^1.

The case of the second action (the trivial one) is more interesting — it gives the full (3-truncated) twisting of differential K-theory. Here, a $\mathscr{G}$-gerbe with connection is:

(a) A surjective submersion $\pi : Y \longrightarrow M$.

(b) A smooth map $f : Y \longrightarrow \mathbb{Z}_2$ and a 2-form $B \in \Omega^2(Y)$.

(c) Over $Y^{[2]}$, the condition that $f \circ \mathsf{pr}_1 = f \circ \mathsf{pr}_2$, further, a smooth map $\phi : Y^{[2]} \longrightarrow \mathbb{Z}_2$ and an S^1-bundle P with connection ω of curvature

$$\mathrm{curv}(\omega) = \mathsf{pr}_2^* B + \phi \cdot (\mathsf{pr}_1^* B).$$

(d) Over $Y^{[3]}$, a connection-preserving bundle gerbe product.

Due to the condition over $Y^{[2]}$, the map $f : Y \longrightarrow \mathbb{Z}_2$ descends to a smooth map $f' : M \longrightarrow \mathbb{Z}_2$ on the base. Thus, a $\mathscr{G}$-gerbe with connection is the same as an $\mathrm{AUT}(S^1)$-gerbe with connection and a independent map $f' : M \longrightarrow \mathbb{Z}_2$.

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