Loop Group Geometry and Transgression

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[1] The talk was invited and prepared, but unfortunately could not take place (the speaker was unavailable)

1.) Gerbes and transgression

2.) Loop group extensions via transgression

3.) Spin structures on loop spaces

Gerbe with connection $\mathcal{G}$ over a smooth manifold M

- cover of M by open sets U_α
- 2-forms $B_\alpha \in \Omega^2(U_\alpha)$
- hermitian line bundles $L_{\alpha\beta}$ over $U_\alpha \cap U_\beta$

  with connection of curvature

$$\mathrm{curv}(L_{\alpha\beta}) = B_\beta - B_\alpha$$

  ($\rightsquigarrow$ curvature $H \in \Omega^3(M)$ of $\mathcal{G}$ defined by $H|_{U_\alpha} = dB_\alpha$)

- connection-preserving isomorphisms

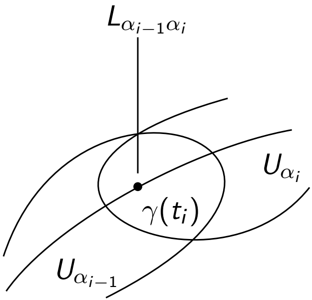
$$\mu_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \longrightarrow L_{\alpha\gamma}$$

  subject to an associativity condition

Define **hermitian line bundle** $L\mathcal{G}$ over loop space $LM = C^\infty(S^1, M)$

- For each loop $\gamma : S^1 \longrightarrow M$, choose $0 = t_0 \leq ... \leq t_n = 1$ and indices $\alpha_1, ..., \alpha_n$ such that

$$\gamma([t_{i-1}, t_i]) \subseteq U_{\alpha_i}$$



- Define the fibre of $L\mathcal{G}$ over γ as

$$L_{\alpha_1\alpha_2|_{\gamma(t_1)}} \otimes ... \otimes L_{\alpha_{n-1}\alpha_n|_{\gamma(t_{n-1})}} \otimes L_{\alpha_n\alpha_1|_{\gamma(t_n)}}$$

Isomorphisms $\mu_{\alpha\beta\gamma} \rightsquigarrow$ independence of n and of indices α_i

Connection on $L_{\alpha\beta} \rightsquigarrow$ independence of $t_i \in \gamma^{-1}(U_{\alpha_i} \cap U_{\alpha_{i+1}})$

Define **connection** $\nu_{\mathcal{G}}$ on $L\mathcal{G}$

- Consider path $\phi : \gamma \longrightarrow \gamma'$ in LM

  ϕ "short" $\rightsquigarrow$ can assume $t_0, \ldots, t_n$ and $\alpha_1, \ldots, \alpha_n$ with

$$\phi([0,1] \times [t_{i-1}, t_i]) \subseteq U_{\alpha_i}$$

- Define parallel transport in $L\mathcal{G}$ by parallel transport in $L_{\alpha_i \alpha_{i+1}}$ along paths $\phi_{t_i} : \gamma(t_i) \longrightarrow \gamma'(t_i)$ in M, with correction by integrals

$$\int_{[0,1]\times[t_{i-1},t_i]} \phi^* B_{\alpha_i}$$

- Identity $\mathrm{curv}(L_{\alpha_i \alpha_{i+1}}) = B_{\alpha_{i+1}} - B_{\alpha_i}$ implies well-definedness

Standard facts (Gawędzki, Brylinski, Murray, Carey, ...):

- Transgression $\mathcal{G} \longmapsto L\mathcal{G}$ is a functor

$$\mathcal{G}rb^{\nabla}(M) \longrightarrow \mathcal{B}un^{\nabla}(LM),$$

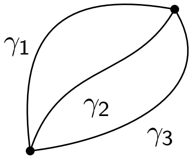
- $\mathrm{curv}(\nu_{\mathcal{G}}) = \int_{S^1} ev^*H$, for $ev : S^1 \times LM \longrightarrow M$
- $c_1(L\mathcal{G}) = \int_{S^1} ev^*DD(\mathcal{G})$, for $DD(\mathcal{G}) \in \mathrm{H}^3(M,\mathbb{Z})$

Recent new facts:

- $L\mathcal{G}$ has a canonical fusion product
- Connection $\nu_{\mathcal{G}}$ is superficial

Fusion product on $L\mathcal{G}$:

- Definition: a **fusion product** on a line bundle P over LM is an associative rule



$$P_{\gamma_1 \cup \gamma_2} \otimes P_{\gamma_2 \cup \gamma_3} \longrightarrow P_{\gamma_1 \cup \gamma_3}$$

where $\gamma_i \cup \gamma_j \in LM$ is obtained by concatenation of γ_i with the inverse of γ_j.

- Technically, a fusion product is a bundle isomorphism over the 3-fold fibre product of $PM \longrightarrow M \times M$.
- For $P = L\mathcal{G}$ the fusion product exists since a point $\gamma_2(t_i)$ on the middle path appears twice, and the contributions cancel:

$$L_{\alpha_i \alpha_{i+1}|_{\gamma_2(t_i)}} \otimes L_{\alpha_{i+1}\alpha_i|_{\gamma_2(t_i)}} \overset{\mu}{\cong} L\alpha_i\alpha_i|_{\gamma_2(t_i)} \overset{\mu}{\cong} \mathbb{C}$$

Superficiality of the connection $\nu_{\mathcal{G}}$ on $L\mathcal{G}$:

- Consider connection ν on a line bundle P over LM. It is called **superficial** if:
  1.) thin loops $\tau \in LLM$ have trivial holonomy: $\mathrm{Hol}_\nu(\tau) = 1$ (thin: $\tau : S^1 \times S^1 \longrightarrow M$ has nowhere full rank)
  2.) thin homotopic loops $\tau, \tau \in LLM$ have the same holonomy: $\mathrm{Hol}_\nu(\tau) = \mathrm{Hol}_\nu(\tau')$
- In order to see that $\nu_{\mathcal{G}}$ on $L\mathcal{G}$ is superficial, one expresses the holonomy of $\nu_{\mathcal{G}}$ as the surface holonomy of the gerbe,

$$\mathrm{Hol}_{\nu_{\mathcal{G}}}(\tau) = \mathrm{Hol}_{\mathcal{G}}(\tau)$$

  and proves that surface holonomy has properties 1.) and 2.).

Summary:

- From every gerbe $\mathcal{G}$ over M one can construct a hermitian line bundle $L\mathcal{G}$ over LM with a fusion product $\lambda_{\mathcal{G}}$ and a superficial connection $\nu_{\mathcal{G}}$.
- Theorem [KW '10]: This gives an equivalence of categories

$$\mathcal{G}rb^{\nabla}(M) \cong \mathcal{F}us\mathcal{B}un^{\nabla_{sf}}(LM).$$

Wait – don't we have to require that the bundles on the right hand side are equivariant under loop rotation?

Fact: equivariance is a consequence of the superficial connection

- Rotation of a loop $\gamma \in LM$ by an angle $\beta \in [0, 2\pi]$ can be regarded as a path in LM,

$$\phi_\beta : \gamma \longrightarrow r_\beta(\gamma)$$

- Get lift $\widetilde{\phi_\beta} : L\mathcal{G}_\gamma \longrightarrow L\mathcal{G}_{r_\beta(\gamma)}$ by parallel transport of $\nu_\mathcal{G}$
- In order to make this an action of S^1, we have to assure

$$\widetilde{\phi_{2\pi}} = \widetilde{\phi_0} = \mathrm{id}.$$

Proof: $\phi_{2\pi}$ is a loop in LM and it is thin.

This argument generalizes to equivariance under $\mathcal{D}\mathit{iff}^+(S^1)$ and under $\mathcal{R}\mathit{ep}^+(S^1)$.

1.) Gerbes and transgression

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The **basic gerbe** $\mathcal{G}_{bas}$ over a compact simple simply-connected Lie group G:

- Uniquely characterized by
  1.) $\mathrm{DD}(\mathcal{G}_{bas})$ generates $\mathrm{H}^3(G,\mathbb{Z}) \cong \mathbb{Z}$
  2.) Curvature H left-invariantly determined by $\langle X, [Y, Z]\rangle$
- Concrete Lie-theoretical construction (Meinrenken '02, Gawędzki-Reis '02)
  - U_α conjugation-invariant, $\alpha = 0, ..., \mathrm{rk}(\mathfrak{g})$
  - $U_\alpha \cap U_\beta \cong \mathcal{O}_{\lambda_\alpha - \lambda_\beta} \subseteq \mathfrak{g}^*$ is a coadjoint orbit, for λ_α vertices of Weyl alcove
  - $L_{\alpha\beta}$ is the prequantum line bundle with Kostant connection

Basic gerbe $\mathcal{G}_{bas}$ is **multiplicative**:

- Isomorphism

$$\mathcal{M} : \mathrm{pr}_1^* \mathcal{G}_{bas} \otimes \mathrm{pr}_2^* \mathcal{G}_{bas} \longrightarrow m^* \mathcal{G}_{bas}$$

  over $G \times G$, with m the product of G.

- Associativity condition over $G \times G \times G$
- Multiplicative structure determines a preimage of $DD(\mathcal{G})$ under the homomorphism

$$\mathrm{H}^4(BG, \mathbb{Z}) \longrightarrow \mathrm{H}^3(G, \mathbb{Z})$$

Apply transgression (first step)

- Obtain Fréchet principal S^1-bundle $L\mathcal{G}_{bas}$ over LG
- Obtain smooth bundle isomorphism

$$L\mathcal{M} : \mathrm{pr}_1^* L\mathcal{G}_{bas} \otimes \mathrm{pr}_2^* L\mathcal{G}_{bas} \longrightarrow Lm^* L\mathcal{G}_{bas}$$

over $LG \times LG$, associative over $LG \times LG \times LG$

Equivalently: a Fréchet Lie group structure on $L\mathcal{G}_{bas}$ making

$$1 \longrightarrow S^1 \longrightarrow L\mathcal{G}_{bas} \longrightarrow LG \longrightarrow 1$$

a central extension

- $c_1(L\mathcal{G}_{bas})$ is the transgression of a generator of $\mathrm{H}^3(G, \mathbb{Z})$
  $\rightsquigarrow$ this is the universal central extension of LG

Apply transgression (second step)

- Obtain connection $\nu_{\mathcal{G}_{bas}}$ on $L\mathcal{G}_{bas}$

  $\rightsquigarrow$ induces splitting s of Lie algebra extension

  $\rightsquigarrow$ induces classifying 2-cocycle $\omega : L\mathfrak{g} \times L\mathfrak{g} \longrightarrow \mathbb{R}$, namely

$$\omega(X, Y) = \int_{S^1} \langle X, Y' \rangle$$

- $\nu_{\mathcal{G}_{bas}}$ is superficial

  $\rightsquigarrow$ canonical equivariance under $\mathcal{D}\mathit{iff}^+(S^1)$ and $\mathcal{R}\mathit{ep}^+(S^1)$

- Remark: $\nu_{\mathcal{G}_{bas}}$ is *not* the standard Pressley-Segal connection

$$\nu_{st} := \theta^{L\mathcal{G}_{bas}} - s(p^*\theta^{LG})$$

  of left-invariant curvature ω. Instead, $\nu_{\mathcal{G}_{bas}} = \nu_{st} + \beta$ for a 1-form $\beta \in \Omega^1(LG)$, and

$$\int_{S^1} \mathrm{ev}^* H = \mathrm{curv}(\nu_{\mathcal{G}_{bas}}) = \omega + d\beta.$$

Apply transgression (third step)

- Obtain fusion product $\lambda_{\mathcal{G}_{bas}}$ on $L\mathcal{G}_{bas}$
- Transgression is a functor:

  $\rightsquigarrow$ fusion product is a group homomorphism
- Fusion product can be seen explicitly in the Mickelsson model

$$\widetilde{LG} = \{(\eta, z) \mid z \in \mathbb{C} \text{ and } \eta : D^2 \longrightarrow G\}/\sim$$

with $(\eta_1, z_1) \sim (\eta_2, z_2)$ if $\eta_1|_{S^1} = \eta_2|_{S^1}$, $z_2 = z_1 e^{2\pi i WZ(\eta_1 \cup \eta_2)}$,
where WZ stands for the Wess-Zumino term.
Namely, for maps $\eta_{ij} : D^2 \longrightarrow G$ with $\eta_{ij}|_{S^1} = \gamma_i \cup \gamma_j$ one has

$$\lambda_{\mathcal{G}}((\eta_{12}, z_{12}) \otimes (\eta_{23}, z_{23})) = (\eta_{13}, z_{12} z_{23} e^{2\pi i WZ(\eta_{12} \cup \eta_{23} \cup \eta_{13})}).$$

Summary: Multiplicative gerbes with connection over G provide models for central extensions of LG with nice properties:

- canonical connections
- canonical equivariant structures
- canonical fusion product

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Motivation from physics:

Supersymmetric field theories suffer from a "global anomaly"

- 1-dimensions: anomaly represented by 2nd Stiefel-Whitney class

  $$w_2 \in \mathrm{H}^2(M, \mathbb{Z}_2)$$

  Cancellation: spin structure on M

- 2-dimensions: anomaly represented by

  $$\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$$

  Cancellation: two approaches:

  1.) Killingback '87: spin structure on LM

  2.) Stolz-Teichner '04: string structure on M

Spin structures on loop spaces (Killingback '87):

- M a spin manifold of dimension n

  $\rightsquigarrow$ frame bundle FM is a $\mathrm{Spin}(n)$-principal bundle

  $\rightsquigarrow$ looped bundle LFM is a Fréchet $L\mathrm{Spin}(n)$-principal bundle

- Definition: a **spin structure** on LM is a lift of the structure group of LFM to the universal central extension

$$1 \longrightarrow S^1 \longrightarrow \widetilde{L\mathrm{Spin}(n)} \longrightarrow L\mathrm{Spin}(n) \longrightarrow 1$$

  i.e. a principal $\widetilde{L\mathrm{Spin}(n)}$-bundle $\widetilde{LFM}$ over LM with an equivariant map $\sigma : \widetilde{LFM} \longrightarrow LFM$.

Obstruction against spin structures on loop spaces:

- Spin structures exists if and only if a certain class

$$\lambda_{LM} \in \mathrm{H}^3(LM, \mathbb{Z})$$

vanishes.

- Theorem [McLaughlin '92]:

$$\lambda_{LM} = \int_{S^1} \mathrm{ev}^* \left(\frac{1}{2} p_1(M) \right)$$

- Thus, we have

$$\frac{1}{2} p_1(M) = 0 \quad \Longrightarrow \quad \lambda_{LM} = 0$$

but the converse is not true in general (Pilch-Warner '88)

$\rightsquigarrow$ we need enhanced notion of spin structures on loop spaces

General lifting theory provides a **reformulation** in terms of principal S^1-bundles and bundle isomorphisms:

- The equivariant map

$$\sigma : \widetilde{LFM} \longrightarrow LFM$$

  exhibits $\widetilde{LFM}$ as a principal S^1-bundle S over LFM.

- The $L\mathrm{Spin}(n)$-action on S can be encoded as an isomorphism

$$\kappa : S \otimes \widetilde{L\mathrm{Spin}(n)} \longrightarrow \rho^* S,$$

  of S^1-bundles over $LFM \times L\mathrm{Spin}(n)$, with ρ the principal action of $L\mathrm{Spin}(n)$ on LFM.

Enhanced version of a spin structure:

- Definition: A **fusion spin structure** on LM is a spin structure (S, κ) with a fusion product λ on the S^1-bundle S over LFM such that

$$\kappa : S \otimes \widetilde{L\mathrm{Spin}(n)} \longrightarrow \rho^* S$$

is fusion-preserving w.r.t. the fusion product $\lambda_{\mathcal{G}_{bas}}$ on $\widetilde{L\mathrm{Spin}(n)} = L\mathcal{G}_{bas}$.

- Theorem [KW '12]: Fusion spin structures exist if and only if

$$\frac{1}{2}p_1(M) = 0$$

Summary:

- The fusion product on the universal central extension $\widetilde{L\mathrm{Spin}(n)}$ allows to define fusion spin structures on loop spaces
- The existence of a fusion spin structure on LM is precisely the condition for the cancellation of the global anomaly of supersymmetric 2-dimensional field theories on M.

Further topics:

- Equivalence between *fusion spin structures on LM* and *string structures on M*

  Kottke-Melrose '13: adding reparameterization invariance

  KW '14: adding thin structures
- Spin connections LM (Coquereaux-Pilch '98)

  KW'14: imposing a superficiality condition makes them equivalent to *string connections* on M
- Anomaly cancellation mechanism: String connections M trivialize a Pfaffian bundle of a family of Dirac operators (Bunke '11)

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