Non-geometric T-duals and non-abelian gerbes

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A **string background** consists of the following data:

- A manifold M ("target space")
- A metric g on M
- A bundle gerbe $\mathcal{G}$ with connection ("B-field")

We recall:

- The **curvature** of $\mathcal{G}$ is a 3-form $H \in \Omega^3(M)$
- The **Dixmier-Douady class** of $\mathcal{G}$ is a class $\xi \in \mathrm{H}^3(M, \mathbb{Z})$
- 2-forms $B \in \Omega^2(M)$ correspond to the case $\xi = 0$
- Neither H nor ξ nor both determine $\mathcal{G}$

Bundle gerbes (with or without connections) form a sheaf of bicategories. They are examples of so-called higher-categorical structure.

This perspective has proved to be very successful, for studying...

- Topological effects, such as discrete torsion and Aharonov-Bohm effects: GawędzkiCarey-Mickelsson-Murray...
- D-branes, in particular in their relation to twisted K-theory: Kapustin, Gawędzki-Reis, Carey-Johnson-Murray...
- Target space description of defect lines and defect networks: Fuchs-Schweigert-W, Runkel-Suszek...
- Orientifolds: Schreiber-Schweigert-W, Gawędzki-Suszek-W, Hekmati-Murray-Szabo-Vozzo...
- Geometric quantization of string backgrounds: Bunk-Szabo, Szabo-Sämann...

T-backgrounds

In toroidal string compactifications, the target space is the total space of a principal $\mathbb{T}^n$-bundle,

$$\begin{array}{c} M = E \circlearrowleft \mathbb{T}^n \\ \big\downarrow \pi \\ X \end{array}$$

T-duality is a relation on the set of toroidal string backgrounds.

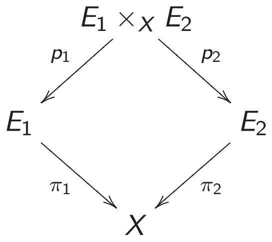
In order to concentrate only on the underlying topology, **T-backgrounds** have been defined as a pair $(E, \mathcal{G})$ of a $\mathbb{T}^n$-principal bundle and a bundle gerbe $\mathcal{G}$ over E; metric and connections are ripped off.

T-duality correspondences

Two T-backgrounds $(E_1, \mathcal{G}_1)$ and $(E_2, \mathcal{G}_2)$ are T-dual, if there exists a **T-duality correspondence**: a bundle gerbe isomorphism

$$\mathcal{D} : p_1^* \mathcal{G}_1 \longrightarrow p_2^* \mathcal{G}_2$$

over the correspondence space



that satisfies the so-called Poincaré condition $\mathcal{P}_x(\mathcal{D})$ for every point $x \in X$.

Remarks:

- Above definition of T-backgrounds and T-duality was coined by Bunke-Rumpf-Schick, based on work by Bouwknegt, Evslin, Hannabuss, Mathai,....
- It is equivalent to an approach via non-commutative topology pursued by Mathai-Rosenberg, Brodzki-Mathai-Rosenberg-Szabo,....
- Including the bundle gerbe is essential: mind the "topology change from H-flux"
- T-duality is symmetric, but neither reflexive nor transitive.
- Particularly interesting for topologists is the existence of a T-duality isomorphism in twisted K-theory,

$$K^*(E_1, \xi_1) \cong K^{*+n}(E_2, \xi_2).$$

T-dualizability

Main question: given a T-background $(E, \mathcal{G})$, does it have any T-duals, and if so, how many?

A complete answer was obtained by Bunke-Rumpf-Schick. The cohomology of E has a filtration

$$\mathrm{H}^3(E, \mathbb{Z}) = F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \cong \mathrm{H}^3(X, \mathbb{Z}).$$

On the level of differential forms, a form $H \in \Omega^3(E)$ is in F_i if it is locally of the form $H = \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_i \wedge ...$, where $x_1, ..., x_i$ are coordinates of X.

We say that a T-background $(E, \mathcal{G})$ is of **class** F_i when i is the biggest number with $\xi_{\mathcal{G}} \in F_i$.

Theorem (Bunke-Rumpf-Schick)

- *A T-background has T-duals if and only if it is of class F_2.*
- *In this case, two T-duals are related by a certain $\mathfrak{so}(n,\mathbb{Z})$-transformation.*

Here, $\mathfrak{so}(n,\mathbb{Z})$ is the additive group of skew-symmetric $(n \times n)$-matrices with integer entries.

For $n = 1$, every T-background is of class F_2, and since $\mathfrak{so}(1,\mathbb{Z}) = \{0\}$, its T-dual is unique. This defines a **T-duality transformation**. Such a transformation does not exist for $n > 1$.

Non-geometric T-folds

If a T-background is only of class F_1 (i.e., locally trivial), it doesn't have any T-duals, they are "mysteriously missing" (Mathai-Rosenberg) or "non-geometric" (Hull).

Non-commutative geometry allows to define these non-geometric T-duals as bundles of non-commutative tori (Mathai-Rosenberg,...).

An example of a T-background in class F_1 is

$$\begin{array}{c} \mathbb{T}^3 = \mathbb{T}^1 \times \mathbb{T}^2 \circlearrowleft \mathbb{T}^2 \\ \downarrow \\ \mathbb{T}^1 \end{array}$$

and over $\mathbb{T}^3$ the bundle gerbe with $\xi = \mathrm{pr}_1^*\gamma \cup \mathrm{pr}_2^*\gamma \cup \mathrm{pr}_3^*\gamma$, where $\gamma \in \mathrm{H}^1(\mathbb{T}^1, \mathbb{Z})$ is a generator.

Higher geometry for non-geometric T-duals

In joint work with Thomas Nikolaus, we propose an alternative treatment of **non-geometric T-duals** in the framework of ordinary (commutative) but higher-categorical geometry.

Our basic observation: every F_1 background is **locally** of class F_2 and so has locally defined T-duals.

We fabricate a new structure we call a **half-geometric T-duality correspondence**. It consists of locally defined T-duals glued together under the $\mathfrak{so}(n, \mathbb{Z})$-transformations.

Our central technique is to use categorical Lie groups as representing objects for sheaves of bicategories.

Categorical Lie groups are the counterparts of ordinary Lie groups and the central objects in higher gauge theory. They can be seen as groups G whose group elements g themselves have gauge symmetries (automorphisms). The corresponding gauge field are non-abelian bundle gerbes of Aschieri-Cantini-Jurco, Schreiber-W, Nikolaus-W.

A simple example is the group $B\mathrm{U}(1)$, where $G = \{e\}$ and the symmetry group of e is $\mathrm{U}(1)$. The bundle gerbe $\mathcal{G}$ in a string background is a $B\mathrm{U}(1)$ gauge field.

Other categorical Lie groups are **central extensions** of ordinary Lie groups G by $B\mathrm{U}(1)$:

$$1 \longrightarrow B\mathrm{U}(1) \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1.$$

They are classified by $\mathrm{H}^4(BG, \mathbb{Z})$.

For example, the famous **String 2-group** is a central extension of $\mathrm{Spin}(n)$, and corresponds to a generator of $\mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z}) = \mathbb{Z}$.

Categorical Lie groups for T-duality

We use **categorical tori** of Ganter, which are central extensions

$$B\mathrm{U}(1) \longrightarrow \mathcal{T}_I \longrightarrow \mathbb{T}^{2n}$$

depending on a symmetric bilinear form $I \in Sym^2(\mathbb{Z}^{2n})$.

The class of $\mathcal{T}_I$ in $\mathrm{H}^4(B\mathbb{T}^{2n}, \mathbb{Z})$ is given by I under the Chern-Weil isomorphism.

Relevant for T-duality is

$$I_n = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix},$$

and we write $\mathbb{TD}_n$ for the categorical torus $\mathcal{T}_{I_n}$.

We prove the following key result:

- Isomorphism classes of $\mathbb{TD}$-gerbes are in bijection with equivalence classes of T-duality correspondences.
- The $\mathfrak{so}(n, \mathbb{Z})$-transformations of Bunke-Rumpf-Schick on T-duality correspondences can be implemented as a strict action by automorphisms of $\mathbb{TD}$.

Then, we perform an abstract construction in higher-categorical geometry: we consider the semi-direct product

$$\mathbb{TD}^{\frac{1}{2}\text{-}geo} := \mathbb{TD} \ltimes \mathfrak{so}(n, \mathbb{Z}).$$

This gives a new categorical Lie group. The corresponding non-abelian bundle gerbes are by definition our **half-geometric T-duality correspondences**.

We prove that half-geometric T-duality correspondences have the following properties:

- The effect of the $\mathfrak{so}(n, \mathbb{Z})$-action on $\mathbb{TD}$ is that the **left leg** of a half-geometric T-duality correspondence is a well-defined T-background of class F_1.
- The **right leg** is not preserved under the action and does not yield any T-background: it is "non-geometric".
- **Every** T-background of class F_1 is the left leg of a **unique** half-geometric T-duality correspondence.

  We see this as a generalized T-duality transformation, valid for any n and all T-backgrounds of class F_1.

Bundle gerbes can be accessed by **local data**. The following is the local data of a half-geometric T-duality:

- transition data for two torus bundles: $a_{ij}, b_{ij} : U_i \cap U_j \longrightarrow \mathbb{R}^n$
- matrices $B_{ij} \in \mathfrak{so}(n, \mathbb{Z})$ satisfying $B_{ik} = B_{ij} + B_{jk}$
- winding numbers for two tori: $n_{ijk}, m_{ijk} \in \mathbb{Z}^n$, with gluing conditions for the tori:

$$
\begin{aligned}
a_{ik} &= n_{ijk} + a_{jk} + a_{ij} \\
b_{ik} &= m_{ijk} + b_{jk} + b_{ij} + B_{jk} a_{ij}
\end{aligned}
$$

  Here we see that the left leg gives a genuine torus bundle, while the gluing of the right leg is spoiled

- transition data for a gerbe: $t_{ijk} : U_i \cap U_j \cap U_k \longrightarrow \mathrm{U}(1)$, subject to a complicated gluing condition depending on the matrices B_{ij}.

Example

Under appropriate choices of sections, one can show that to the T-background

$$(\mathbb{T}^3, \xi = \mathrm{pr}_1^*\gamma \cup \mathrm{pr}_2^*\gamma \cup \mathrm{pr}_3^*\gamma)$$

over $\mathbb{T}^1$ corresponds the half-geometric T-duality correspondence with all local data trivial except for the matrices B_{ij}, whose non-trivial entries satisfy

$$\gamma = [B_{ij}^{12}] = -[B_{ij}^{21}] \in \check{\mathrm{H}}^1(\mathbb{T}^1, \mathbb{Z}).$$

Remarks: T-duality group

We also compute the (higher) automorphism group $\mathrm{Aut}(\mathbb{TD})$ and show that

$$\pi_0(\mathrm{Aut}(\mathbb{TD})) = \mathrm{O}^{\pm}(n, n, \mathbb{Z})$$

This group contains the split-orthogonal group $\mathrm{O}(n, n, \mathbb{Z})$ as a subgroup of index two. It appeared already in work of Mathai-Rosenberg.

One can regard $\mathfrak{so}(n, \mathbb{Z})$ as a subgroup of $\mathrm{O}(n, n, \mathbb{Z})$, and we prove that $\mathrm{Aut}(\mathbb{TD})$ splits canonically over this subgroup. Our action of $\mathfrak{so}(n, \mathbb{Z})$ on $\mathbb{TD}$ is induced via this splitting.

Remarks: T-folds

Our half-geometric T-duality correspondences can be seen as a baby version of Hull's T-folds, in terms of the doubled-geometry perspective of Hull.

Indeed, the matrices B_{ij} of a half-geometric T-duality correspondence form a globally defined (and non-trivial) $\mathfrak{so}(n, \mathbb{Z})$-bundle over X. A "polarization" would be a local trivialization of that bundle.

Under such a trivialization, the half-geometric T-duality correspondence reduces to an ordinary T-duality correspondence between the globally defined left leg and a locally defined right leg. Its correspondence space is a locally defined $\mathbb{T}^{2n}$-principal bundle; that's the doubled geometry.

Summary

- Topological T-duality correspondences only exist between T-backgrounds of class F_2.
- Half-geometric T-duality correspondences exist between left legs of class F_1 and non-geometric right legs.
- There is a generalized T-duality transformation: every T-background of class F_1 can be extended in a unique way to a half-geometric T-duality correspondence.
- Our treatment of half-geometric T-duality correspondences explores new examples of categorical Lie groups and their associated non-abelian gerbes.

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