

Introduction to  
Higher Parallel Transport

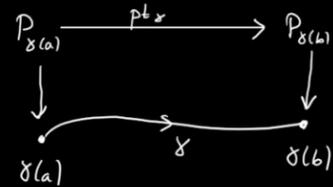
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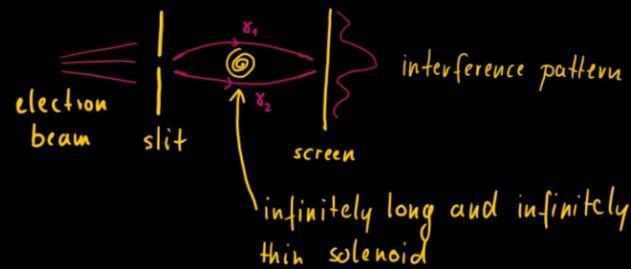
# Ordinary Parallel Transport

## Basic setup for parallel transport

- Manifold  $X$
- Fibre bundle  $P$  over  $X$  with a "connection"
- Path  $\gamma: [a,b] \rightarrow X$
- Parallel transport along  $\gamma$ :



## Aharonov-Bohm-effect



Classical Maxwell theory:

electromagnetic field outside of solenoid is zero ( $F=0$ )

Experiment:

interference pattern depends on current in the solenoid!

Textbook answer:

Gauge potential matters (= parallel transport matters:  $pt_{\gamma_1} \neq pt_{\gamma_2}$ )

$\Leftrightarrow$  non-trivial holonomy around the solenoid, depending on current

## Example 1

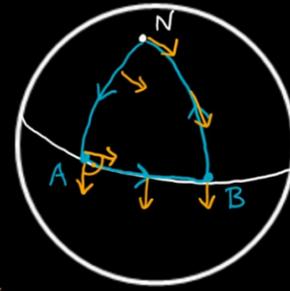
$$P = TS^2$$

Connection = metric

$\gamma$  closed curve  
"holonomy"

$$pt_\gamma: T_A S^2 \rightarrow T_A S^2$$

non-trivial ( $\Rightarrow$  curvature)



## Local description of connections ( $G = U(1)$ )

- $P$  principal  $U(1)$ -bundle
- $\{U_\alpha\}$  covering of  $X$  by open sets
- $g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow U(1)$  transition functions

$$g_{\beta\gamma} \cdot g_{\alpha\beta} = g_{\alpha\gamma}$$

A connection is a family  $\{A_\alpha\}$  of 1-forms,

$$A_\alpha \in \Omega^1(U_\alpha) \quad \text{"local gauge potentials"}$$

such that

$$A_\beta = A_\alpha + \frac{1}{i} g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

"gauge transformations on overlaps"

$$\text{Curvature: } F \in \Omega^2(X), \quad F|_{U_\alpha} = dA_\alpha$$

"Field strength"

Aharonov-Bohm effect  $\Leftrightarrow F=0$ , but  $A_\alpha \neq 0$ .

## Example 2 Gauge theory

Particles with "internal" states (phase, spin, ...)

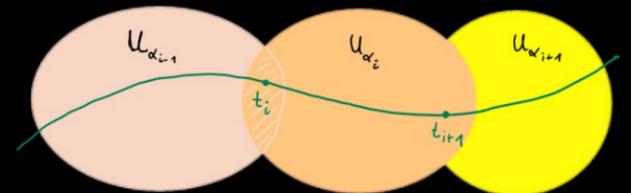
( $\Rightarrow$  "internal" symmetry group  $G$ )



Parallel transport describes change of internal state:

$$pt_\gamma(\Psi(a)) = \Psi(b)$$

## Local description of holonomy



$$\text{Hol}(\gamma) = \prod_i \exp\left(i \int_{t_i}^{t_{i+1}} A_{\alpha_i} d\gamma\right) \cdot \prod_i g_{\alpha_{i-1}\alpha_i}(\gamma(t_i)) \in U(1)$$

usual action functional for a particle in a gauge potential

## Path integral

Integrand of the particle in a gauge potential:

$$\mathcal{A}(\gamma) := e^{iS_{\text{kin}}(\gamma)} \cdot \text{Hol}(\gamma)$$

Aharonov-Bohm effect:  $\text{Hol}(\gamma) \neq 1$

# Higher Parallel Transport (Higher Gauge Theory)



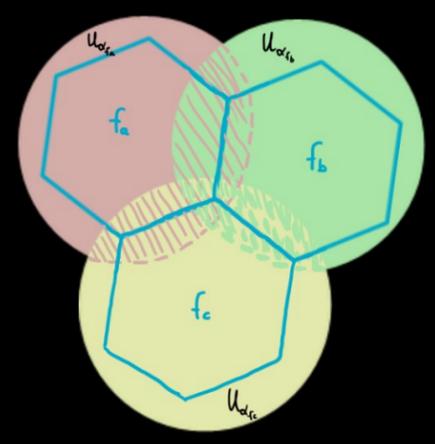
2d surface, closed and oriented  
"world sheet" of a string

Consider a smooth map  $\phi: \Sigma \rightarrow X$ .

Goal: define a 2d holonomy  $\text{Hol}(\phi) \in U(1)$ .

use in the path integral of the string ("Wess-Zumino-term")

Idea: generalize the formula for ordinary holonomy:



$$\text{Hol}(\phi) := \prod_f \exp\left(i \int_f \phi^* B_{\alpha_f}\right) \cdot \prod_{e \in \partial f} \exp\left(i \int_e \phi^* A_{\alpha_f \alpha_e}\right) \cdot \prod_{v \in \partial e} g_{\alpha_f \alpha_e \alpha_v}(\phi(v))$$

[Alvarez '85, Gawędzki '86]

Need: 2-forms  $B_\alpha \in \Omega^2(U_\alpha)$   
1-forms  $A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta)$   
functions  $g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \rightarrow U(1)$

For well-definedness, this has to satisfy:

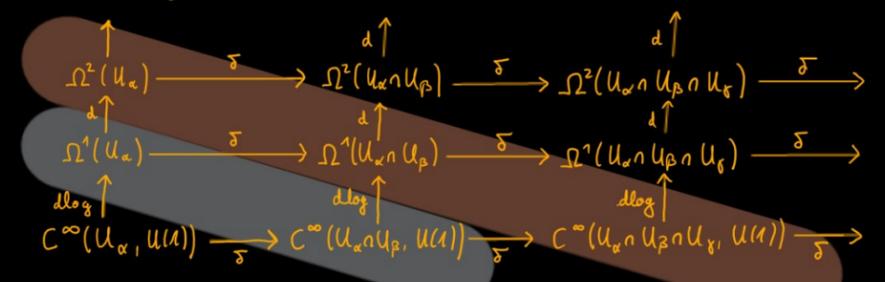
$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

$$A_\beta + A_\alpha = A_\beta + \frac{1}{i} g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

$$g_{\beta\gamma\alpha} \cdot g_{\alpha\beta\gamma} = g_{\alpha\beta\gamma} \cdot g_{\alpha\beta\gamma}$$

## Deligne cohomology

Čech-Deligne double complex:



... works in any higher degree!

data for 1d holonomy      data for 2d holonomy

Upshot: Deligne cohomology  
:= total cohomology of the Čech-Deligne double complex\* in degree n  
≅ data for n-dimensional holonomy.

\* the complex needs to be truncated above, otherwise we only get flat things.

[Gawędzki '86, Brylinski '93]

Many open questions left at this point:

- Higher holonomy for non-abelian groups instead of  $U(1)$
  - Higher parallel transport instead of holonomy
- solved by transport functor formalism  
[Baez-Schreiber '04, Schreiber-KW '08]



- What are the "geometric objects" behind higher degree Deligne cohomology classes?
  - connections on bundle gerbes [Murray '95]
  - connections on principal 2-bundles [Baez-Bartels '04, Wockel '11]
- New phenomena:
  - surfaces can be non-orientable  $\implies$  orientifolds [Schreiber-Schweigert-KW '05]
  - surfaces can have "physical boundary"  $\implies$  D-branes [Gawędzki '00, Carey et al. '02]

The toolbox "Homological algebra" can now be used.  
The following sequences are exact:

$$0 \longrightarrow H^n(X, U(1)) \longrightarrow \hat{H}^n(X) \longrightarrow \Omega_{d, \mathbb{Z}}^{n+1}(X) \longrightarrow 0$$

↑ "discrete torsion"      ↑ Deligne cohomology      ↑ curvature

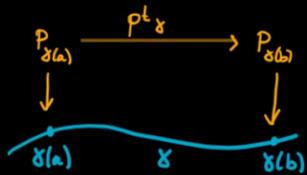
$$0 \longrightarrow \Omega^n(X) / d\Omega^{n-1}(X) \longrightarrow \hat{H}^n(X) \longrightarrow H^{n+1}(X, \mathbb{Z}) \longrightarrow 0$$

↓ global gauge potentials      ↓ "Chern class"

# Transport functor formalism

Main idea: axiomatize the properties of parallel transport

1st step: go back to ordinary parallel transport



- Properties:
- 1.) if  $\gamma$  is a constant path, then  $pt_\gamma = id$ .
  - 2.) if two paths  $\gamma_1, \gamma_2$  are concatenable, then  $pt_{\gamma_2 * \gamma_1} = pt_{\gamma_2} \circ pt_{\gamma_1}$ .
  - 3.)  $pt_\gamma$  depends only on the thin homotopy class of  $\gamma$ .
  - 4.)  $pt_\gamma$  depends on  $\gamma$  in a locally smooth way.

parallel transport is a functor  
concerns the domain of the functor  
locally smooth functor

[Barrett '91]

[Caetano-Picken '93]

Thin homotopy

$h$  is a fixed-ends-homotopy  
"without area": its image is 1-dimensional.

Example 1:

Example 2: any reparameterization

Path groupoid of  $X$ :  $\mathcal{P}_1(X)$

- objects: points of  $X$
- morphisms: thin homotopy classes of paths in  $X$

Gauge groupoid of  $G$ :  $BG$

- just a single object  $*$
- each group element is a morphism  $* \xrightarrow{g} *$

Theorem (Schreiber - KW '07)

$$\text{Fun}^\infty(\mathcal{P}_1(X), BG) \cong \text{Con}_G(X)$$

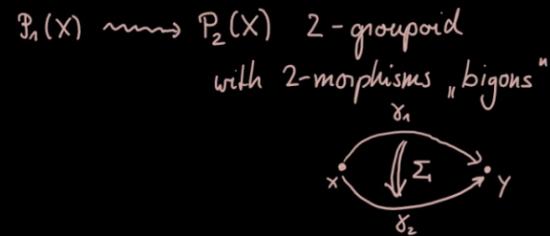
- objects: gauge potentials  $A \in \Omega^1(X, \mathfrak{g})$
- morphisms: gauge transformations

$$[\gamma \mapsto \text{Pexp} \int_\gamma A] \longleftrightarrow A$$

Localization / Stabilization  
Adding some information about the fibres

$$\text{Trans}(\mathcal{P}_1(X), G\text{-Tor}) \cong \text{Bun}_G^\nabla(X)$$

Next: categorify this!



$BG \rightsquigarrow$  Lie 2-groupoid with a single object ("Lie 2-group", "smooth crossed module")

Smooth crossed module

2 Lie groups  $H, G$   
a Lie group homomorphism  $t: H \rightarrow G$   
an action  $h \mapsto {}^h h$  of  $G$  on  $H$  s.t.

- 1.)  $t({}^h h) = gt(h)g^{-1}$
- 2.)  $t(h^{-1})h = h'h^{-1}$

Example 1:  $U(1) \rightarrow \{e\}$ ,  
this gives the abelian case discussed before

Example 2:  $H \xrightarrow{i} \text{Aut}(H)$   
 $i(h)(h') = h'h^{-1}$   
 $\Psi_h = \Psi(h)$

Example 3: String 2-group  
Example 4: T-duality 2-group

Theorem (Schreiber - KW '09) "Form follows functor"

$$\text{Fun}^\infty(\mathcal{P}_2(X), B\Gamma)$$

- objects:  $(A, B), A \in \Omega^1(X, \mathfrak{g}), B \in \Omega^2(X, \mathfrak{g})$
- 1-morphisms...  $t_*(B) = dA + \frac{1}{2}[A, A]$
- 2-morphisms... "fake flatness"

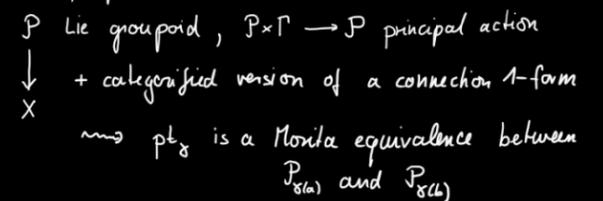
Localization / Stabilization

$$\text{Trans}(\mathcal{P}_2(X), B(H \xrightarrow{t} G)) \cong \text{Non-abelian Čech-Deligne cocycles}$$

[Breen-Messing '98]

explains fake-flatness  
equips Čech-Deligne cocycles with a parallel transport

Global perspective: Principal 2-bundles



Theorem [KW '17]:  $\text{Trans}(\mathcal{P}_2(X), \Gamma\text{-Tor}) \cong \text{Bun}_\Gamma^\nabla(X)$ .