The 2-Category of Bundle Gerbes

von

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich diese Arbeit ohne fremde Hilfe selbstständig verfasst und nur die angegebenen Hilfsmittel benutzt habe.

Konrad Waldorf

Contents

"La Gerbe", Henri Matisse.

Introduction

Klassische Eichtheorien punktförmiger Teilchen – wie zum Beispiel die Theorie der Elektrodynamik – kann man als Theorie von Vektorbündeln mit Zusammenhang verstehen. Dabei entspricht die Feldstärke des Eichfeldes der Krümmung des Zusammenhangs, und die Wirkung eines Teilchens, das sich durch das Eichfeld bewegt, wirdzu einem Teil durch Paralleltransport entlang seiner Weltlinie bestimmt.

Man unterscheidet zwei Arten von Weltlinien, geschlossene und solche mit Enden. Der Paralleltransport um eine geschlossene Weltlinie läßt sich als Holonomie des Zusammenhangs ausdrücken; diese nimmt Werte in der Strukturgruppe des Vektorbündels an. In dieser Arbeit wird das immer $U(1)$ sein. Im Fall einer Weltlinie mit Enden werden an den Endpunkten Trivialisierungen des Vektorbündels gewählt, so dass der Paralleltransport um solche Kurven ebenfalls Werte in dieser Gruppe annimmt. Ich werde deshalb vereinheitlichend in beiden Fällen nur noch von Holonomie sprechen.

Ebenso wichtig wie Vektorbündel mit Zusammenhang selbst sind Morphismen von Vektorbündeln mit Zusammenhang: sie bringen Eichtheorien, die durch verschiedene Vektorbündel mit Zusammenhang definiert werden, in Verbindung und ermöglichen eine Klassifizierung von Eichtheorien. Es ist demnach natürlich, Vektorbündel mit Zusammenhang als Kategorie zu behandeln.

Während die gerade erwähnten, punktförmigen (null-dimensionalen) Teilchen eindimensionale Weltlinien besitzen, beschäftigt man sich in der Stringtheorie mit eindimensionalen physikalischen Objekten, die zweidimensionale Weltflächen überstreichen. Man erwartet, dass sich eine Eichtheorie für solche Teilchen in gewissem Sinne analog zu der oben skizzierten Eichtheorie für punktförmige Teilchen verhält. Zu ihrer Beschreibung wird also zunächst ein Objekt benötigt, welches die Berechnung von Holonomie entlang einer zweidimensionalen Fläche erlaubt, und somit die Rolle des Vektorbündels mit Zusammenhang in der Stringtheorie übernimmt. Ein derartiges Objekt wird allgemein als Gerbe mit Zusammenhang bezeichnet.

Für den Begriff Gerbe existieren in der Literatur verschiedene, teilweise nichtäquivalente Definitionen [Bry93, Mur96, Hit01, Moe02], von denen aber nur einige geeignet zu sein scheinen, zur Definition einer Gerbe mit Zusammenhang ausgedehnt zu werden. Eine davon ist Gegenstand dieser Arbeit: die Bündelgerbe. Da ich vor allem an ihrer Holonomie interessiert bin, werde ich in dieser Arbeit von vorneherein nur Bündelgerben mit Zusammenhang behandeln, und diese von nun an verkürzt als Bündelgerben bezeichnen.

Zunächst wurden Bündelgerben als alleinstehende Objekte eingeführt [Mur96]. Um sie zu klassifizieren, wurden später Morphismen von Bündelgerben definiert [MS00], sogenannte stabile Isomorphismen. Durch sie wird eine Äquivalenzrelation auf der Menge der Bündelgerben definiert, deren Äquivalenzklassen in Bijektion zur Deligne-Kohomologiegruppe $\mathrm{H}^2(M,\mathcal{D}_2)$ stehen. Im Umgang mit stabilen Isomorphismen stellt es sich jedoch heraus [Ste00, SSW05], dass man hier – im Gegensatz zu Morphismen in einer Kategorie – zum Vergleich zweier stabiler Isomorphismen einen Begriff von Morphismen von stabilen Isomorphismen benötigte. Damit ist klar, dass Bündelgerben keine Kategorie bilden können. Vielmehr wird in [Ste00] gezeigt, dass Bündelgerben zusammen mit stabilen Isomorphismen und Morphismen von stabilen Isomorphismen – sogenannten Transformationen – ein 2-Groupoid[1] bilden.

Der Nutzen von Bündelgerben in der Stringtheorie und in der Lagrange'schen Formulierung von zweidimensionaler konformer Feldtheorie ist erwiesen [GR02, Gaw05, SSW05], wird aber hier nur eine untergeordnete Rolle spielen. Man kann diesen Anwendungen aber die Motivation für die Untersuchung zweier Begriffe entnehmen: der einer Trivialisierung und der eines Gerbenmoduls einer Bündelgerbe. Um nämlich die Holonomie einer Bündelgerbe entlang einer Weltfläche zu berechnen, sind zunächst – analog zur Unterscheidung zwischen geschlossenen Weltlinien und solchen mit Enden bei punktförmigen Teilchen – geschlossene Weltflächen und solche mit Rand zu unterscheiden. Im Fall einer Weltfläche mit Rand werden durch einen Gerbenmodul Randbedingungen gestellt. Dann wird in beiden Fällen die Holonomie durch die Wahl einer Trivialisierung berechnet.

In dieser Arbeit schlage ich eine Definition für eine neue Art von Morphismen zwischen Bündelgerben vor, die ich stabile Morphismen[2] nenne (Definition 1.3a). Entsprechend definiere ich Morphismen zwischen stabilen Morphismen (Definition 1.5a). Eine gewisse Klasse der stabilen Morphismen definiere ich als stabile Isomorphismen (Definition 1.3b). Als Eigenschaften dieser Definitionen möchte ich drei Erkenntnisse darlegen.

Erstens ergibt sich, dass die Äquivalenzrelation, die durch die hier definierten stabilen Isomorphismen auf der Menge der Bündelgerben definiert wird, mit der oben erwähnten Äquivalenzrelation aus [MS00] übereinstimmt; der Isomorphiebegriff von Bündelgerben bleibt also unberührt. Ich liefere den Beweis dieser Aussage im Appendix (Theorem A.1).

Zweitens umfasst die hier gegebene Definition von stabilen Morphismen vereinheitlichend Trivialisierungen und Gerbenmoduln: eine Trivialisierung einer Bündelgerbe $\mathcal{G}$ ist demnach ein stabiler Isomorphismus von $\mathcal{G}$ in eine bestimmte Art trivialer Bündelgerbe, und ein Gerbemodul ist ein stabiler Morphismus von $\mathcal{G}$ in eine solche triviale Bündelgerbe. Diese Aussagen sind, zusammen mit den grundlegenden Definitionen einer Bündelgerbe, eines stabilen Morphismus, und eines Morphismus von stabilen Morphismen, Gegenstand von Kapitel 1 dieser Arbeit.

Als drittes Resultat zeige ich in Kapitel 2, dass Bündelgerben zusammen mit den hier definierten stabilen Morphismen und Morphismen von stabilen Morphismen ei-

[1] In [Ste00] wird zwar von einem Bigroupoid gesprochen, dessen Definition stimmt jedoch mit der hier gegebenen Definition einer 2-Kategorie (mit zusätzlichen Eigenschaften) überein.

[2] Dieses Wort wird zwar auch in [Ste00] verwendet, aber in einem anderen Sinn.

ne 2-Kategorie $\mathfrak{BGrb}(M)$ bilden. Dazu stelle ich ausführlich die Struktur zusammen, und überprüfe die Axiome. Es stellt sich heraus, dass die so definierte 2-Kategorie der Bündelgerben ist nicht strikt ist (Remark 2.4A). Weiter ergibt sich, dass die stabilen Isomorphismen genau die invertierbaren 1-Morphismen von $\mathfrak{BGrb}(M)$ sind (Theorem 2.5A), so dass im Nachhinein der Begriff stabiler Isomorphismus gerechtfertigt wird.

Die Definition der 2-Kategorie $\mathfrak{BGrb}(M)$ der Bündelgerben über M erlaubt einen präzisen Umgang mit Bündelgerben und ihren Morphismen: im letzten Abschnitt dieser Arbeit gebe ich an, wie die Wohldefiniertheit der Definitionen von Holonomie von Bündelgerben und Holonomie von Bündelgerben mit Gerbenmoduln – so wie sie in [CJM02] gegeben wurden – in direkter Weise aus den Eigenschaften der 2-Kategorie $\mathfrak{BGrb}(M)$ folgt, und nicht aufwändig bewiesen werden muss.

Mit der Definition von $\mathfrak{BGrb}(M)$ wird unmittelbar deutlich, dass die Eichtransformationen einer Bündelgerbe $\mathcal{G}$ – das sind alle stabilen Morphismen von $\mathcal{G}$ nach $\mathcal{G}$ – eine Kategorie bilden: das ist nämlich in jeder 2-Kategorie der Fall. Sie bietet außerdem die Möglichkeit, den vermuteten Zusammenhang von Bündelgerben mit einer gewissen noch zu spezifizierenden Klasse von 2-Bündeln präzise als Äquivalenz von 2-Kategorien zu formulieren.

Ich bedanke mich sehr herzlich bei meinen Gutachtern, Christoph Schweigert und Birgit Richter. Außerdem bei Urs Schreiber, der meine Aufmerksamkeit auf die 2-kategoriellen Aspekte der Theorie von Bündelgerben gelenkt hat, und mit dem ich unzählige Diskussionen zu diesem Thema führen durfte. Letztendlich wäre diese Arbeit aber ohne die Fachbereichsverwaltung des Departments Mathematik nicht zustande gekommen; ebenfalls nicht ohne die wohlwollende Unterstützung durch den Studiendekan, Herrn Professor Bodo Werner.

1 Bundle Gerbes

The first section 1.1 introduces the language of categories and descent theory, which will be used throughout this thesis. After the well-known definition of a bundle gerbe in section 1.2, I present my definition of stable morphisms and stable isomorphisms of bundle gerbes in section 1.3, together with their law of composition. In section 1.4 I show that it contains trivializations and bundle gerbe modules of a bundle gerbe as special cases. The last section provides the definition of morphisms of stable morphisms.

1.1 The monoidal Stack of Vector Bundles

The theory of bundle gerbes is build on the theory of (complex) vector bundles over smooth manifolds and their descent theory. In many situations the following question arises: when A is a vector bundle over a smooth manifold Y, and $\pi : Y \to M$ is a smooth map, is there another vector bundle B over M, such that π^*B and A are isomorphic vector bundles?

Because the same question also arises for morphisms of vector bundles, it is natural to discuss it in a categorial framework. The discussion leads to the definition of the stack of vector bundles.

Just to be complete, I start with the definition of a (small) category.

Definition 1.1a. *A category* $\mathfrak{C}$ *consists of the following data:*

- *a set of objects* $\mathrm{Obj}(\mathfrak{C})$*,*
- *for each two objects* $\mathcal{A}, \mathcal{A}'$ *a set of morphisms* $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}, \mathcal{A}')$*,*
- *for each object* $\mathcal{A}$ *an identity morphism* $\mathrm{id}_{\mathcal{A}} \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}, \mathcal{A})$ *and*
- *for each three objects* $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ *a map*

$$\circ \;:\; \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}', \mathcal{A}'') \times \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}, \mathcal{A}') \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}, \mathcal{A}''), \tag{1}$$

such that the following two axioms are satisfied:

(C1) Identity: for each morphism $\beta \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}, \mathcal{A}')$ *the identity axiom*

$$\mathrm{id}_{\mathcal{A}'} \circ \beta = \beta = \beta \circ \mathrm{id}_{\mathcal{A}}. \tag{2}$$

(C2) Associativity: for all morphisms $\beta \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}_1, \mathcal{A}_2)$, $\beta' \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}_2, \mathcal{A}_3)$ *and* $\beta'' \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}_3, \mathcal{A}_4)$ *the associativity axiom*

$$(\beta'' \circ \beta') \circ \beta = \beta'' \circ (\beta' \circ \beta). \tag{3}$$

If $\mathfrak{C}$ and $\mathfrak{D}$ are two categories, one can form the product category $\mathfrak{C} \times \mathfrak{D}$ with objects $\mathrm{Obj}(\mathfrak{C}) \times \mathrm{Obj}(\mathfrak{D})$, morphisms $\mathrm{Hom}_{\mathfrak{C}\times\mathfrak{D}}((X,Y),(X',Y')) = \mathrm{Hom}_{\mathfrak{C}}(X,X') \times \mathrm{Hom}_{\mathfrak{D}}(Y,Y')$ and the composition is also the direct product of the composition maps from $\mathfrak{C}$ and $\mathfrak{D}$. A trivial example of a category is the category $\mathtt{1}$ consisting of only one object $\bullet$ and with $\mathrm{Hom}(\bullet,\bullet) = \{\mathrm{id}_\bullet\}$. For this category, I canonically identify the direct products $\mathtt{1} \times \mathfrak{C}$ and $\mathfrak{C} \times \mathtt{1}$ for any category $\mathfrak{C}$ with $\mathfrak{C}$ itself for simplicity.

An invertible morphism $\beta \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{A}, \mathcal{A}')$ is called isomorphism, and the category $\mathfrak{C}$ is called a groupoid, if every morphism is an isomorphism. Recall further that

- a functor $F : \mathfrak{C} \to \mathfrak{D}$ consists of maps $F : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Obj}(\mathfrak{D})$ and $F : \mathrm{Hom}_{\mathfrak{C}}(X,Y) \to \mathrm{Hom}_{\mathfrak{D}}(F(X), F(Y))$ respecting the composition of morphisms and the identity morphism. For each category $\mathfrak{C}$ there is the functor $\mathrm{id}_{\mathfrak{C}}$ consisting of identity maps.
- a natural transformation $\eta : F \Rightarrow F'$ of two functors $F, F' : \mathfrak{C} \to \mathfrak{D}$ is for each object $X \in \mathrm{Obj}(\mathfrak{C})$ a morphism $\eta_X \in \mathrm{Hom}_{\mathfrak{C}}(F(X), F'(X))$, such that for every morphism $f \in \mathrm{Hom}_{\mathfrak{C}}(X,Y)$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & F'(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F'(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & F'(Y) \end{array} \tag{4}$$

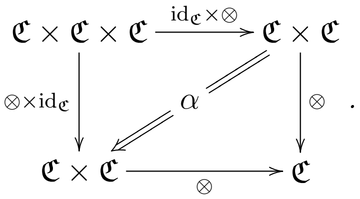
  commutes. For each functor F there is the identity natural transformation $\mathrm{id} : F \Rightarrow F$. A natural transformation is called natural equivalence, if the morphism η_X is an isomorphism for each object X.
- a functor $F : \mathfrak{C} \to \mathfrak{D}$ is called an equivalence of categories, if there is a functor $F^* : \mathfrak{D} \to \mathfrak{C}$ and two natural equivalences

$$l \ : \ F \circ F^* \Longrightarrow \mathrm{id}_{\mathfrak{D}} \tag{5}$$
$$r \ : \ F^* \circ F \Longrightarrow \mathrm{id}_{\mathfrak{C}}. \tag{6}$$

As a first example of a category, let $\mathfrak{Bun}(M)$ be the category of vector bundles over a smooth manifold M, together with morphisms of vector bundles, and the usual composition of morphisms of vector bundles. Throughout this thesis, by vector bundle I mean a smooth hermitian complex vector bundle of finite rank with connection, and all morphisms of vector bundles are meant to be smooth and to preserve the hermitian metric and the connection. The preservation of the hermitian metric implies that every morphism is an isomorphism, hence $\mathfrak{Bun}(M)$ is even a groupoid.

DEFINITION 1.1B. *A monoidal category is a category* $\mathfrak{C}$ *together with the following additional structure:*

- *a functor* $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ *together with a natural equivalence*[3]



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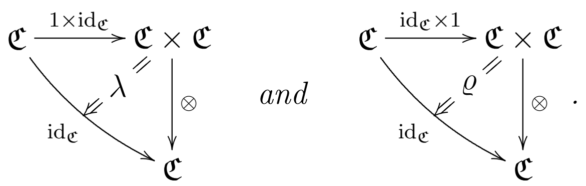
- *a functor* $1 : \mathbf{1} \to \mathfrak{C}$, *together with natural equivalences*



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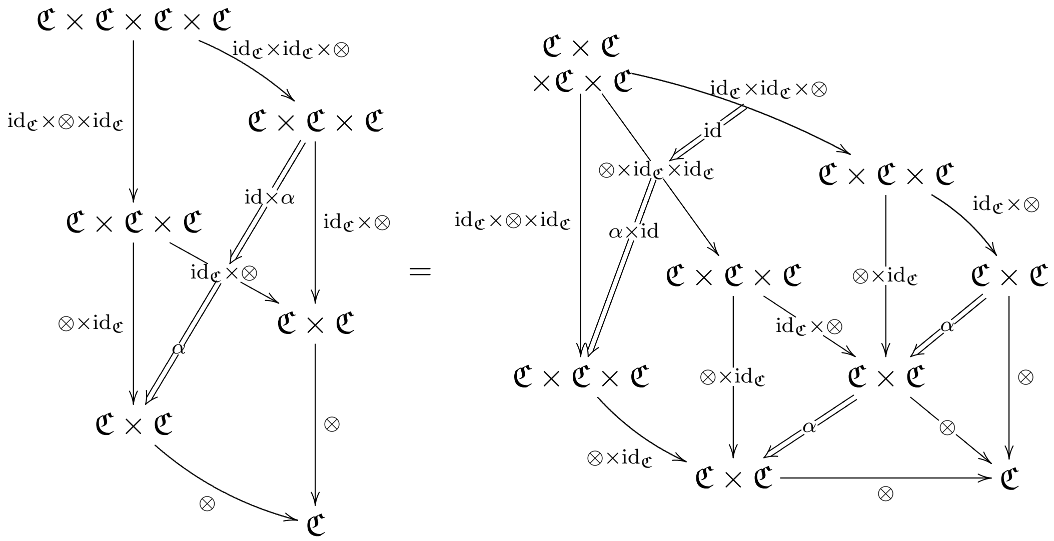
The three natural equivalences have to satisfy the following two coherence axioms:

(MC1) The Pentagon identity[4]*:*



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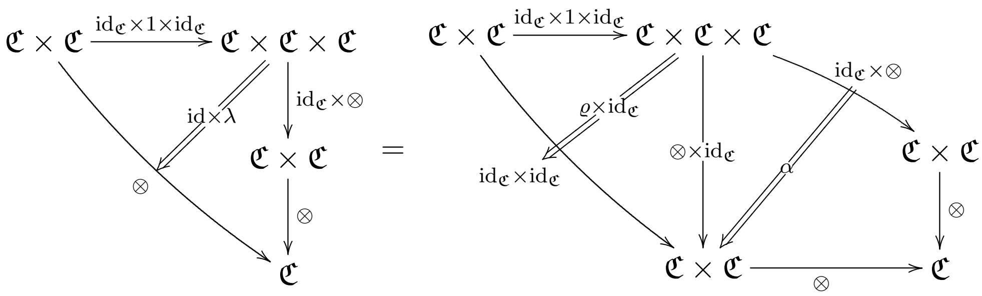
[3]Diagrams of this type are used to express, that α is a natural equivalence from the functor $\otimes \circ (\mathrm{id}_{\mathfrak{C}} \times \otimes)$ to the functor $\otimes \circ (\otimes \times \mathrm{id}_{\mathfrak{C}})$ – which are both functors from $\mathfrak{C} \times \mathfrak{C} \times \mathfrak{C}$ to $\mathfrak{C}$– with a view to avoid the composition symbol $\circ$.

[4]Equations of two diagrams like the next one are equations of natural equivalences: parsing the boundary arrows clockwise, one obtains from both diagrams the same functor, here $\otimes \circ (\mathrm{id}_{\mathfrak{C}} \times \otimes) \circ (\mathrm{id}_{\mathfrak{C}} \times \mathrm{id}_{\mathfrak{C}} \times \otimes)$. By parsing counter-clockwise, one obtains another functor, here $\otimes \circ (\otimes \times \mathrm{id}_{\mathfrak{C}}) \circ (\mathrm{id}_{\mathfrak{C}} \times \otimes \times \mathrm{id}_{\mathfrak{C}})$. These two functors are both functors from $\mathfrak{C} \times \mathfrak{C} \times \mathfrak{C} \times \mathfrak{C}$ to $\mathfrak{C}$, and each diagram is a natural transformation between the clockwise parsed functor and the counter-clockwise one. Now the equation states that the two natural transformations are equal.

(MC2) The triangle identity:



According to the definitions of functors and natural transformations there are obvious definitions of monoidal functors and monoidal natural transformations, which can be found in the literature [ML97].

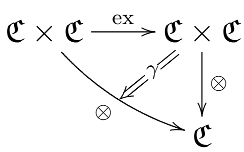
Recall that I presented the groupoid $\mathfrak{Bun}(M)$ of vector bundles over a smooth manifold M as an example of a category. Now I endow this groupoid with the structure of a monoidal category. I denote the trivial vector bundle $M \times \mathbb{C}$ (equipped with the trivial flat connection) by 1. To make the groupoid $\mathfrak{Bun}(M)$ together with the trivial vector bundle 1 and the usual tensor product of vector bundles into a monoidal category, I just have to note that the natural equivalences α, λ are given by the canonical isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C \quad (9)$$
$$\lambda_A : 1 \otimes A \longrightarrow A \quad (10)$$
$$\varrho_A : A \otimes 1 \longrightarrow A \quad (11)$$

of vector bundles over M. As known from the theory of vector bundles, these isomorphisms are coherent in the sense of the axioms (MC1) and (MC2) of a monoidal category. In this thesis I will often abbreviate these canonical isomorphisms by "$\cong$", to simplify equations and diagrams, and only sometimes use α, λ_A and ϱ_A to emphasize their presence.

In the following, two additional structures on monoidal categories are important.

- Let $\mathrm{ex} : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ be the functor which exchanges the two components of a direct product. A symmetry on a monoidal category $(\mathfrak{C}, 1, \otimes, \alpha, \lambda, \varrho)$ is a natural equivalence



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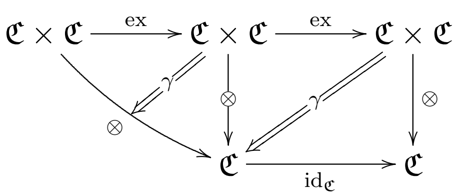
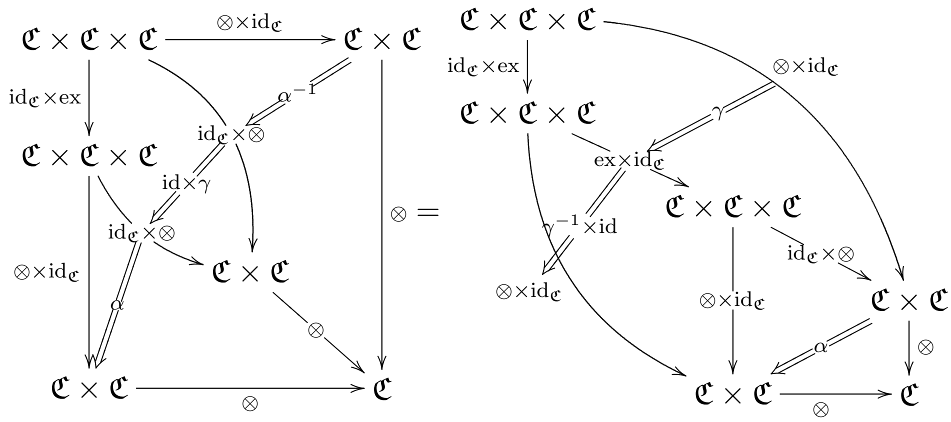
such that

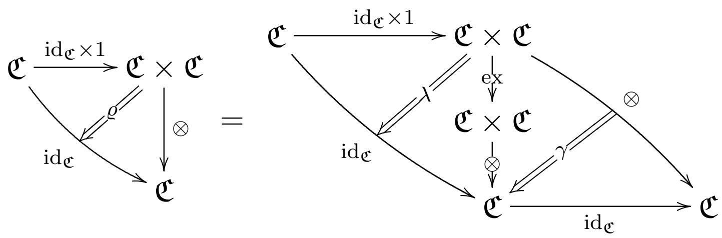


$= \mathrm{id}_\otimes$,

and which is compatible with α,



and with λ and ϱ:



A monoidal category with a symmetry is called symmetric monoidal category.

- Suppose there is a monoidal category $\mathfrak{C}(M)$ for each smooth manifold M. A pullback structure on $\mathfrak{C}(M)$ is a functor

$$f^* \;:\; \mathfrak{C}(M) \;\longrightarrow\; \mathfrak{C}(N) \tag{13}$$

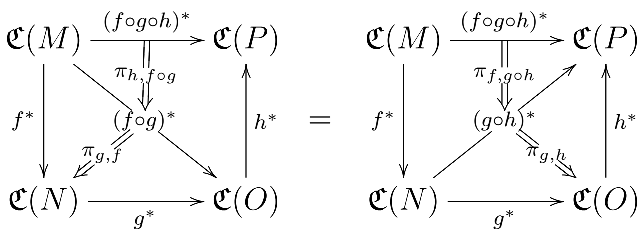
for each smooth map $f : N \to M$ together with a natural equivalence

$$\pi_{g,f} \;:\; (f \circ g)^* \;\Longrightarrow\; g^* \circ f^*, \tag{14}$$

for each two composable maps $f : N \to M$ and $g : O \to N$ which is coherent for threefold compositions

$$P \xrightarrow{h} O \xrightarrow{g} N \xrightarrow{f} M \tag{15}$$

of smooth maps, which means that



(16)

as natural transformations from $(f \circ g \circ h)^*$ to $h^* \circ g^* \circ f^*$, and which is compatible with the monoidal structures (I don't write out the diagrams here).

The groupoid $\mathfrak{Bun}(M)$ comes naturally with a symmetry and a pullback structure: the symmetry is given by the canonical isomorphisms

$$\gamma_{A,B} \;:\; A \otimes B \longrightarrow B \otimes A \tag{17}$$

for each two vector bundles A and B. To define its pullback structure, recall [Bar04], that the pullback of a vector bundle A via a smooth map $f : N \to M$ is uniquely defined up to canonical isomorphisms of vector bundles. This gives a functor $f^* : \mathfrak{Bun}(M) \to \mathfrak{Bun}(N)$, sending the bundle A over M to the fibre product $f^*A := N \times_M A$, and an isomorphism $\varphi : A \to B$ of vector bundles over M to the isomorphism $f^*\varphi := 1 \times \varphi$ of vector bundles over N. Note that the pullback is compatible with the composition of smooth maps $f : N \to M$ and $g : O \to N$ in the sense that there are canonical natural equivalences

$$(\pi_{g,f})_A \;:\; (f \circ g)^* A \Longrightarrow g^* f^* A, \tag{18}$$

which are coherent when concerned with a threefold composition, and compatible with the other structure of $\mathfrak{Bun}(M)$.

Now let $\pi : Y \to M$ be a surjective submersion. Whenever I am concerned with a surjective submersion, by $Y^{[k]}$ I denote the k-fold fibre product of Y with itself, which is again a smooth manifold. I denote the canonical projections from $Y^{[k]}$ to $Y^{[l]}$ with $l < k$ by $\pi_{i_1 \dots i_l}$, where the indices indicate those components of $Y^{[k]}$ on which is projected. Note that whenever $f : Y_1 \to Y_2$ is a fibre-preserving map between surjective submersions, there are canonically induced maps $f : Y_1^{[k]} \to Y_2^{[k]}$ on the k-fold fibre products, which I denote by the same letter for simplicity.

A morphism of surjective submersions $\pi : Y \to M$ and $\pi' : Y' \to M$ is a surjective submersion $p : Y \to Y'$ such that $\pi = \pi' \circ p$. Here I want to think of a surjective submersion $\pi : Y \to M$ as a generalization of an open cover of M. Recall [Bar04], that open covers are in bijection to surjective local diffeomorphisms $\pi : Y \to M$, where for an open cover $\{U_\alpha\}_{\alpha \in A}$ one sets

$$Y := \bigsqcup_{\alpha \in A} U_\alpha, \tag{19}$$

and π is the inclusion $\pi : U_\alpha \hookrightarrow M$. To develop the theory of bundle gerbes, local diffeomorphism is generalized to submersion. This generalization is important for the construction of concrete examples of bundle gerbes [Mei02, GR03] (cf. section 1.2).

Corresponding to the generalization of open covers to surjective submersions, I introduce a generalization of a stack over a smooth manifold defined on open covers [Moe02], to the definition of a stack over a smooth manifold defined on surjective submersions.

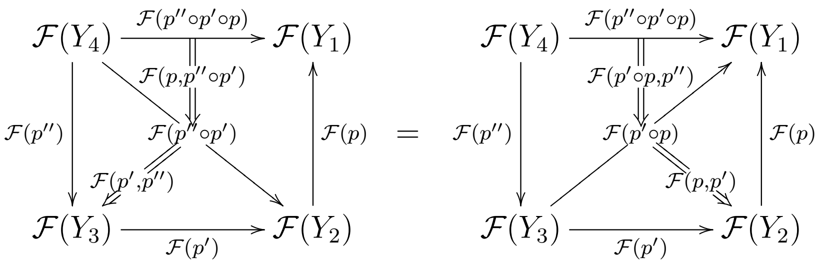
DEFINITION 1.1C. *A fibred category $\mathcal{F}$ over M consists of*

- *a category $\mathcal{F}(Y)$ for each surjective submersion $\pi : Y \to M$.*
- *a functor $\mathcal{F}(p) : \mathcal{F}(Y_2) \to \mathcal{F}(Y_1)$ for each morphism $p : Y_1 \to Y_2$ of surjective submersions.*

- *a natural equivalence*

$$\mathcal{F}(p,p') \;:\; \mathcal{F}(p'\circ p) \implies \mathcal{F}(p)\circ\mathcal{F}(p') \tag{20}$$

for each pair $p : Y_1 \to Y_2$, $p' : Y_2 \to Y_3$ of composable morphisms of surjective submersions, such that for three composable morphisms of surjective submersions p, p' and $p'' : Y_3 \to Y_4$ the equality



(21)

of natural transformations holds.

To formulate the gluing axiom, define a category $\mathfrak{Des}(\mathcal{F},p)$ of descent data for a given fibred category $\mathcal{F}$ and any morphism $p : Y \to Y'$ of surjective submersions as follows:

- its objects are pairs (A,α) where A is an object of the category $\mathcal{F}(Y)$ and $\alpha : \mathcal{F}(p_1)(A) \to \mathcal{F}(p_2)(A)$ is a morphism in the category $\mathcal{F}(Y^{[2]})$ such that

$$\mathcal{F}(p_{11})(\alpha) \;=\; \mathrm{id}_{\mathcal{F}(p_1)(A)} \tag{22}$$
$$\mathcal{F}(p_{13})(\alpha) \;=\; \mathcal{F}(p_{23})(\alpha)\circ\mathcal{F}(p_{12})(\alpha) \tag{23}$$

  as morphisms in the category $\mathcal{F}(Y^{[3]})$.
- a morphism $\beta : (A,\alpha) \Rightarrow (A',\alpha')$ is a morphism $\beta : A \to A'$ in the category $\mathcal{F}(Y)$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(p_1)(A) & \xrightarrow{\mathcal{F}(p_1)(\beta)} & \mathcal{F}(p_1)(A') \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathcal{F}(p_2)(A) & \xrightarrow[\mathcal{F}(p_2)(\beta)]{} & \mathcal{F}(p_2)(A') \end{array} \tag{24}$$

  of morphisms in the category $\mathcal{F}(Y^{[2]})$ commutes.
- the composition of morphisms is just the composition of morphisms in $\mathcal{F}(Y)$.

Note that the pullback along p defines a canonical functor $D_p : \mathcal{F}(Y') \to \mathfrak{Des}(\mathcal{F},p)$. The following definition of a stack is analogous to the definition of a stack defined on open covers [Moe02].

DEFINITION 1.1D. *The fibred category $\mathcal{F}$ is called a stack, provided D_p is an equivalence of categories for each morphism p.*

Now I want to give a concrete example of a stack, namely the stack of vector bundles over a given smooth manifold M. So I define a fibred category $\mathcal{B}$ over M as follows. For a surjective submersion $\pi : Y \to M$ let $\mathcal{B}(Y) := \mathfrak{Bun}(Y)$. For a morphism of surjective submersions $p : Y_1 \to Y_2$ let $\mathcal{B}(p) := p^*$. The natural equivalences $\mathcal{B}(p, p') : (p' \circ p)^* \to p^* \circ p'^*$ are given by the natural equivalence $\pi_{p,p'}$ for pullbacks of vector bundles. The coherence of these natural equivalences shows that $\mathcal{B}$ is a fibred category. I already mentioned that the categories $\mathfrak{Bun}(Y)$ are groupoids. For such "fibred groupoids" equation (22) follows from (23) by the existence of inverses.

Theorem 1.1e. *The fibred category $\mathcal{B}$ is a stack.*

Proof . Consider the descent category $\mathfrak{Des}(\mathcal{B}, p)$ for a morphism $p : Y \to Y'$ of surjective submersions. By definition, an object (A, α) is a vector bundle $A \to Y$ together with an isomorphism

$$\alpha \;:\; \pi_1^* A \;\longrightarrow\; \pi_2^* A \tag{25}$$

of vector bundles over $Y^{[2]}$, which satisfies the cocycle condition

$$\pi_{13}^*\alpha = \pi_{23}^*\alpha \circ \pi_{12}^*\alpha \tag{26}$$

over $Y^{[3]}$. A morphism in $\mathfrak{Des}(\mathcal{B}, p)$ is a morphism

$$\beta : A \;\longrightarrow\; A' \tag{27}$$

of vector bundles over Y, which is compatible with α and α' in the sense that the diagram

$$\begin{array}{ccc} \pi_1^* A & \xrightarrow{\alpha} & \pi_2^* A \\ {\scriptstyle \pi_1^*\beta}\downarrow & & \downarrow{\scriptstyle \pi_2^*\beta} \\ \pi_1^* A' & \xrightarrow[\alpha']{} & \pi_2^* A' \end{array} \tag{28}$$

of morphisms of vector bundles over $Y^{[2]}$ commutes. The canonical functor D_p is the functor $p^* : \mathfrak{Bun}(Y') \to \mathfrak{Des}(\mathcal{B}, p)$, which maps a vector bundle $L \to Y'$ to (p^*L, id), and a morphism $\beta : L_1 \to L_2$ of vector bundles over Y' to $p^*\beta$.

The category $\mathfrak{Des}(\mathcal{B}, p)$ consists exactly of those vector bundles, which satisfy the well-known descent condition for bundles [Bry93]. This defines a functor $D_p^* : \mathfrak{Des}(\mathcal{B}, p) \to \mathfrak{Bun}(Y')$, such that $D_p^* \circ D_p$ and $D_p \circ D_p^*$ are naturally equivalent to the respective identity functors. The functor D_p^* can be written down explicitly, but I don't need the construction in the rest of this thesis. □

The definition of a fibred category extends naturally to the definition of a fibred symmetric monoidal category. This way it is clear that the stack $\mathcal{B}$ is a symmetric monoidal stack.

1.2 Bundle Gerbes

By line bundle I mean a vector bundle of rank 1; according to my conventions this is a hermitian complex vector bundle of rank 1 with connection. Line bundles over M form a full subcategory $\mathfrak{Lin}(M)$ of $\mathfrak{Bun}(M)$. In the same way I defined the stack $\mathcal{B}$ of vector bundles, there is the symmetric monoidal stack $\mathcal{LB}$ of line bundles.

Definition 1.2a. *A bundle gerbe $\mathcal{G}$ over a smooth manifold M consists of the following data:*

- *a surjective submersion $\pi : Y \to M$ which is called the covering of $\mathcal{G}$,*
- *a line bundle $L \to Y^{[2]}$,*
- *a 2-form $C \in \Omega^2(Y)$, which is called the curving of $\mathcal{G}$, and*
- *an isomorphism*

$$\mu \;:\; \pi_{12}^*L \otimes \pi_{23}^*L \;\longrightarrow\; \pi_{13}^*L \tag{29}$$

  of line bundles over $Y^{[3]}$

such that the following two axioms are satisfied:

(G1) *the curvature of L is compatible with the curving C, i.e.*

$$\mathrm{curv}(L) = \pi_2^*C - \pi_1^*C \tag{30}$$

(G2) *μ is associative in the sense that the diagram*

$$\begin{array}{ccc} \pi_{123}^*(\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_{34}^*L & \xrightarrow{\pi_{123}^*\mu\otimes 1} & \pi_{123}^*\pi_{13}^*L \otimes \pi_{34}^*L \\ \cong\| & & \|\cong \\ \pi_{12}^*L \otimes \pi_{234}^*(\pi_{12}^*L \otimes \pi_{23}^*L) & & \pi_{134}^*(\pi_{12}^*L \otimes \pi_{23}^*L) \\ {\scriptstyle 1\otimes\pi_{234}^*\mu}\downarrow & & \downarrow{\scriptstyle \pi_{134}^*\mu} \\ \pi_{12}^*L \otimes \pi_{234}^*\pi_{13}^*L & & \pi_{134}^*\pi_{13}^*L \\ \cong\| & & \|\cong \\ \pi_{124}^*(\pi_{12}^*L \otimes \pi_{23}^*L) & \xrightarrow[\pi_{124}^*\mu]{} & \pi_{124}^*\pi_{13}^*L \end{array} \tag{31}$$

of isomorphisms of line bundles over $Y^{[4]}$ commutes.

In the diagram, by $\cong$ I indicate for simplification the usage of the canonical natural equivalences α, γ and π of the symmetric monoidal category $\mathfrak{Lin}(Y^{[4]})$.

To every bundle gerbe $\mathcal{G}$ over M one assigns a 3-form $\mathrm{curv}(\mathcal{G}) \in \Omega^3(M)$, which is called the curvature of the bundle gerbe. To this end, observe that from axiom (G1) it follows that $\pi_1^*\mathrm{d}C = \pi_2^*\mathrm{d}C$, since the curvature of a line bundle is a closed form. Hence, there is a unique 3-form $H \in \Omega^3(M)$ with the property $\pi^*H = \mathrm{d}C$. Define this 3-form to be the curvature $\mathrm{curv}(\mathcal{G}) := H$ of the bundle gerbe $\mathcal{G}$.

To give at least one concrete example of a bundle gerbe, I construct to a given 2-form $\varrho \in \Omega^2(M)$ on a smooth manifold M a bundle gerbe $\mathcal{I}_\varrho$, which I call the canonical bundle gerbe with B-field ϱ. To this end, let the covering be $Y := M$ with $\pi := \mathrm{id}_M$, so that $Y^{[2]} \cong M$. Let the line bundle L be the trivial line bundle, and the isomorphism μ be the identity. The curving is $C := \varrho \in \Omega^2(Y)$. Now the axiom (G1) for gerbe data is satisfied, since $\mathrm{curv}(L) = 0$ and $\pi_1 = \pi_2 = \mathrm{id}_M$. The associativity axiom (G2) is trivially satisfied, so I have defined a bundle gerbe $\mathcal{I}_\varrho$. For the curvature of this bundle gerbe one finds $\mathrm{curv}(\mathcal{I}_\varrho) = \mathrm{d}\varrho$.

Other examples of bundle gerbes are constructed in [GR02, Mei02, GR03]. Especially the construction of bundle gerbes over simply-connected Lie groups G different from $SU(N)$ or $Sp(4n)$ shows, that the generalization of open covers to surjective submersions mentioned in section 1.1 is essential: here Y is not the disjoint union of the sets of an open cover of G. Constructions for other manifestations of gerbes with connection – such as Deligne cocycles or Hitchin gerbes [Hit01] – for such Lie groups are not known.

For a bundle gerbe $\mathcal{G}$, a consequence of the existence of the isomorphism μ is that the line bundle L restricted to the image of the diagonal map $\Delta_{11} : Y \to Y^2$ is trivial. Furthermore, I construct an isomorphism $t_\mu : \Delta_{11}^*L \to 1$ of line bundles over Y via the pullback of the isomorphism μ along the diagonal map $\Delta_{111} : Y \to Y^{[3]}$,

$$\Delta_{11}^*L \cong (\Delta_{11}^*L \otimes \Delta_{11}^*L) \otimes \Delta_{11}^*L^* \xrightarrow{\Delta_{111}^*\mu\otimes 1} \Delta_{11}^*L \otimes \Delta_{11}^*L^* \cong 1. \tag{32}$$

Here $\cong$ again indicates the canonical natural equivalences of $\mathfrak{Lin}(Y)$. I denote the degeneracy maps of $Y^{[k]}$ by $\Delta_{i_1,\dots,i_k} : Y^{[l]} \to Y^{[k]}$, so that Δ_{11} and Δ_{111} reproduce the above diagonal maps. The isomorphism t_μ is characterized by the following

Proposition 1.2b. *The isomorphism $t_\mu : \Delta_{11}^*L \to 1$ has the properties*

$$\varrho_L \circ (1 \otimes \pi_2^* t_\mu) = \Delta_{122}^*\mu \quad \textit{and} \quad \lambda_L \circ (\pi_1^* t_\mu \otimes 1) = \Delta_{112}^*\mu, \tag{33}$$

where ϱ_L and λ_L are the natural equivalences of the monoidal category $\mathfrak{Bun}(Y^{[2]})$.

Proof. This is a direct consequence of the associativity axiom (G2) for the isomorphism μ, which gives the equations

$$1 \otimes \pi_2^*\Delta_{111}^*\mu = \Delta_{122}^*\mu \otimes 1 \quad \text{and} \quad \pi_1^*\Delta_{111}^*\mu \otimes 1 = 1 \otimes \Delta_{112}^*\mu \tag{34}$$

of isomorphisms of line bundles over $Y^{[2]}$. □

1.3 Stable Morphisms and their Composition

In this section $\mathcal{G}_1$ and $\mathcal{G}_2$ are two bundle gerbes over M. I denote the data of both bundle gerbes by the same letters as in Definition 1.2a but with indices 1 or 2 respectively. For the following definition, recall that all vector bundles are complex hermitian vector bundles with connection.

Definition 1.3a. *A stable morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ of bundle gerbes is*

- *a surjective submersion $\zeta : Z \to M$ which is called the covering of $\mathcal{A}$,*

- *two maps $y_i : Z \to Y_i$ for $i = 1, 2$ such that the diagram*

$$\begin{array}{ccc} Z & \xrightarrow{y_2} & Y_2 \\ {\scriptstyle y_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ Y_1 & \xrightarrow[\pi_1]{} & M \end{array} \tag{35}$$

  commutes,
- *a vector bundle $A \to Z$ of rank n and*
- *an isomorphism*

$$\alpha \;:\; y_1^*L_1 \otimes \zeta_2^*A \;\longrightarrow\; \zeta_1^*A \otimes y_2^*L_2 \tag{36}$$

  of vector bundles over $Z^{[2]}$

such that two axioms are satisfied, namely

(SM1) the curvature of A is a real 2-form and fixed by

$$\mathrm{curv}(A) = y_2^*C_2 - y_1^*C_1 \tag{37}$$

(SM2) the isomorphism α commutes with the isomorphisms μ_1 and μ_2 of the gerbes in the sense that the diagram

$$\begin{array}{ccc}
y_1^*(\pi_{12}^*L_1 \otimes \pi_{23}^*L_1) \otimes \zeta_3^*A & \xrightarrow{y_1^*\mu_1 \otimes \mathrm{id}} & y_1^*\pi_{13}^*L_1 \otimes \zeta_3^*A \\
\cong\| & & \|\cong \\
\zeta_{12}^*y_1^*L_1 \otimes \zeta_{23}^*(y_1^*L_1 \otimes \zeta_2^*A) & & \zeta_{13}^*(y_1^*L_1 \otimes \zeta_2^*A) \\
{\scriptstyle 1 \otimes \zeta_{23}^*\alpha}\big\downarrow & & \big\downarrow \\
\zeta_{12}^*y_1^*L_1 \otimes \zeta_{23}^*(\zeta_1^*A \otimes y_2^*L_2) & & \big\downarrow{\scriptstyle \zeta_{13}^*\alpha} \\
\cong\| & & \big\downarrow \\
\zeta_{12}^*(y_1^*L_1 \otimes \zeta_2^*A) \otimes \zeta_{23}^*y_2^*L_2 & & \big\downarrow \\
{\scriptstyle \zeta_{12}^*\alpha \otimes 1}\big\downarrow & & \big\downarrow \\
\zeta_{12}^*(\zeta_1^*A \otimes y_2^*L_2) \otimes \zeta_{23}^*y_2^*L_2 & & \zeta_{13}^*(\zeta_1^*A \otimes y_2^*L_2) \\
\cong\| & & \|\cong \\
\zeta_1^*A \otimes y_2^*(\pi_{12}^*L_2 \otimes \pi_{23}^*L_2) & \xrightarrow[1 \otimes y_2^*\mu_2]{} & \zeta_1^*A \otimes y_2^*\pi_{13}^*L_2
\end{array} \tag{38}$$

of isomorphisms of vector bundles over $Z^{[3]}$ commutes.

Several remarks are in order:

(1) The existence of the isomorphism α imposes a condition for the curvature of the vector bundle A. The condition (SM1) on the curvature is compatible with this condition.

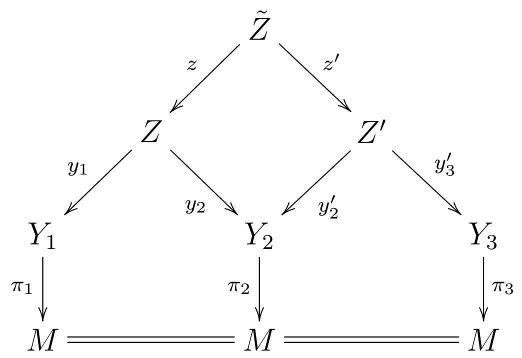
(2) There is always a canonical choice for the covering Z by taking the fibre product $Z := Y_1 \times_M Y_2$ of the coverings of the bundle gerbes. Restricted to this choice of the covering, and to vector bundles of rank 1, Definition 1.3B coincides with the definitions of stable isomorphisms in [Ste00, CJM02, GR02, SSW05].

(3) In the Appendix it is shown that admitting other coverings than the fibre product is not an essential generalization, which means that any stable isomorphism is "equivalent" to another one whose covering is the fibre product.

Definition 1.3b. *A stable morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ is called stable isomorphism, if the vector bundle A is a line bundle, i.e. has rank 1.*

I will show in section 2.5, that the term isomorphism is justified in the sense that stable isomorphisms as defined here are exactly those stable morphisms which are invertible in a certain sense.

An important feature of stable morphisms is that they can be composed. Already when defining the composition it turns out that admitting general coverings for stable isomorphisms simplifies calculations a lot compared to the composition in [Ste00]. Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ be three bundle gerbes over M, and let $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ be two stable morphisms. Consider the fibre product $\tilde{Z} := Z \times_{Y_2} Z'$, i.e. the commutative diagram



(39)

The space $\tilde{Z}$ comes with a canonical surjective submersion $\tilde{\xi} : \tilde{Z} \to M$.

Definition 1.3c. *The composition of $\mathcal{A}$ with $\mathcal{A}'$ is the stable isomorphism*

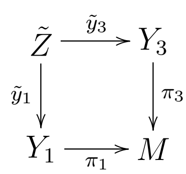
$$\mathcal{A}' \circ \mathcal{A} \;:\; \mathcal{G}_1 \longrightarrow \mathcal{G}_3, \tag{40}$$

consisting of the covering $\tilde{\xi} : \tilde{Z} \to M$, together with the two projections $\tilde{y}_1 := y_1 \circ z$ and $\tilde{y}_3 := y_3' \circ z'$, the line bundle $\tilde{A} := z^*A \otimes z'^*A'$ over $\tilde{Z}$ and the isomorphism

$$\tilde{\alpha} := (\mathrm{id} \otimes z'^*\alpha') \circ (z^*\alpha \otimes \mathrm{id}) \tag{41}$$

of line bundles over $\tilde{Z}^{[2]}$.

It is easy to verify that everything is well-defined: the diagram



(42)

is the outer bound of diagram (39), and hence commutes. For the curvature of $\tilde{A}$ we find

$$\begin{aligned} \mathrm{curv}(\tilde{A}) &= z^*\mathrm{curv}(A) + z'^*\mathrm{curv}(A') && (43)\\ &= z^*y_2^*C_2 - z^*y_1^*C_1 + z'^*y_3'^*C_3 - z'^*y_2'^*C_2 && (44)\\ &= \tilde{y}_3^*C_3 - \tilde{y}_1^*C_1 && (45)\end{aligned}$$

so that axiom (SM1) is satisfied. To see that the composition (41) gives the correct isomorphism

$$\tilde{\alpha} \;:\; \tilde{y}_1^*L_1 \otimes \tilde{\zeta}_2^*\tilde{A} \;\longrightarrow\; \tilde{\zeta}_1^*\tilde{A} \otimes \tilde{y}_3'^*L_3, \tag{46}$$

first notice that I suppressed three occurrences of the canonical natural equivalence α of the monoidal category $\mathfrak{Bun}(\tilde{Z}^{[2]})$. The extended version of (41) is

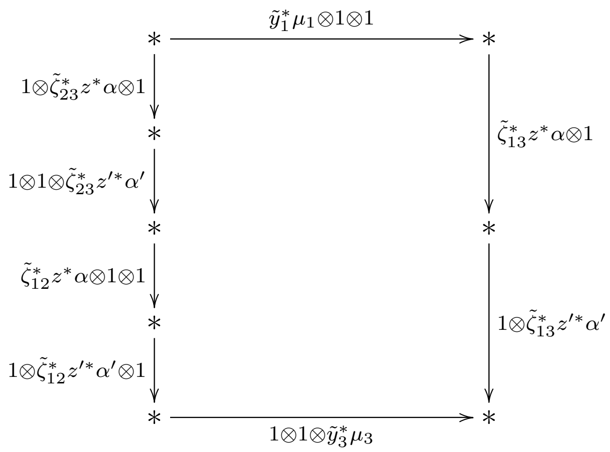
$$\tilde{\alpha} := \alpha_{A,A',L_3} \circ (\mathrm{id} \otimes z'^*\alpha') \circ \alpha^{-1}_{A,L_2,A'} \circ (z^*\alpha \otimes \mathrm{id}) \circ \alpha_{L_1,A,A'}, \tag{47}$$

where I dropped the pullbacks in the indices of α for simplicity. The natural equivalences arrange the bracketing of tensor products such that the composition of the maps

$$\begin{aligned} z^*\alpha \otimes \mathrm{id} : (\tilde{y}_1^*L_1 \otimes z^*\zeta_2^*A) \otimes z'^*\zeta_2'^*A' &\;\to\; (z^*\zeta_1^*A \otimes z^*y_2^*L_2) \otimes z'^*\zeta_2'^*A' && (48)\\ \mathrm{id} \otimes z'^*\alpha' : z^*\zeta_1^*A \otimes (z'^*y_2^*L_2 \otimes z'^*\zeta_2'^*A') &\;\to\; z^*\zeta_1^*A \otimes (z'^*\zeta_1'^*A' \otimes \tilde{y}_3^*L_3) && (49)\end{aligned}$$

gives the correct isomorphism (46).

Axiom (SM2) for this isomorphism is the commutativity of the diagram (I omit the expressions at the vertices for simplicity)



(50)

of isomorphisms of vector bundles over $\tilde{Z}$. To prove its commutativity, note that the isomorphisms $\tilde{\zeta}_{23}^*z'^*\alpha'$ and $\tilde{\zeta}_{12}^*z^*\alpha$ on the left hand side of the diagram act independently on different tensor factors, so that one can permute their order. After that, the diagram splits horizontally:

$$\begin{array}{ccc}
* & \xrightarrow{z^*y_1^*\mu_1\otimes 1\otimes 1} & * \\
{\scriptstyle 1\otimes\tilde{\zeta}_{23}^* z^*\alpha\otimes 1}\downarrow & & \\
* & & \downarrow{\scriptstyle \tilde{\zeta}_{13}^* z^*\alpha\otimes 1} \\
{\scriptstyle \tilde{\zeta}_{12}^* z^*\alpha\otimes 1\otimes 1}\downarrow & & \\
* & \xrightarrow[z'^*y_2'^*\mu_2]{z^*y_2^*\mu_2} & * \\
{\scriptstyle 1\otimes 1\otimes\tilde{\zeta}_{23}^* z'^*\alpha'}\downarrow & & \\
* & & \downarrow{\scriptstyle 1\otimes\tilde{\zeta}_{13}^* z'^*\alpha'} \\
{\scriptstyle 1\otimes\tilde{\zeta}_{12}^* z'^*\alpha'\otimes 1}\downarrow & & \\
* & \xrightarrow[1\otimes 1\otimes z'^*y_3'^*\mu_3]{} & *
\end{array} \tag{51}$$

The upper half diagram is the pullback of axiom (SI2) for the isomorphism α by z, and the lower half diagram is the pullback of axiom (SI2) for α' by z'. So the two half diagrams commute. Both labels at the middle arrow coincide, so that the outer bound diagram commutes, too.

An immediate consequence of Definition 1.3c is, that the composition of stable morphisms restricts to a composition of stable isomorphisms:

Corollary 1.3d. *If $\mathcal{A}$ and $\mathcal{A}'$ are stable isomorphisms, also the composed stable morphism $\mathcal{A}' \circ \mathcal{A}$ is a stable isomorphism.*

1.4 Trivializations and Bundle Gerbe Modules

Recall the definition of the canonical bundle gerbe $\mathcal{I}_\varrho$ with B-field ϱ. Then the following definition is quite natural:

Definition 1.4a. *A trivialization of a bundle gerbe $\mathcal{G}$ is a 2-form $\varrho \in \Omega^2(M)$ together with a stable isomorphism $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\varrho$. A bundle gerbe $\mathcal{G}$ is called topologically trivial, if it admits a trivialization. It is called trivial, if it admits a trivialization with $\varrho = 0$.*

To show, that this definition reproduces the usual definition of a trivialization [SSW05, GR02, CJM02], I write out the details: a stable isomorphism $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\varrho$ consists by definition of a line bundle $T \to Z$. Assume that the covering Z is chosen canonically, which in this particular situation amounts to $Z = Y \times_M M \cong Y$. So T is a line bundle over Y. The projection from Z to Y is here the identity map, and the projection from Z to M is the surjective submersion $\pi : Y \to M$ of the bundle gerbe $\mathcal{G}$. The trivialization $\mathcal{T}$ further consists of an isomorphism of line bundles, which simplifies here to an isomorphism

$$\tau \;:\; L \otimes \pi_2^* T \;\longrightarrow\; \pi_1^* T \tag{52}$$

of line bundles over $Y^{[2]}$. The axiom (SM2) for a stable morphism is the compatibility of the isomorphism τ with the isomorphism μ of $\mathcal{G}$, and reduces here to:

$$\begin{array}{ccc}
\pi_{12}^*L \otimes \pi_{23}^*L \otimes \pi_3^*T & \xrightarrow{\mu\otimes \mathrm{id}} & \pi_{13}^*L \otimes \pi_3^*T \\
\cong \| & & \| \cong \\
\pi_{12}^*L \otimes \pi_{23}^*(L \otimes \pi_2^*T) & & \pi_{13}^*(L \otimes \pi_2^*T) \\
\downarrow {\scriptstyle 1\otimes\pi_{23}^*\tau} & & \downarrow {\scriptstyle \pi_{13}^*\tau} \\
\pi_{12}^*L \otimes \pi_{23}^*\pi_1^*T & & \pi_{13}^*\pi_1^*T \\
\cong \| & & \| \cong \\
\pi_{12}^*(L \otimes \pi_2^*T) & \xrightarrow[\pi_{12}^*\tau]{} & \pi_{12}^*\pi_1^*T
\end{array} \tag{53}$$

Equation (52) and diagram (53) are exactly the conditions one demands usually for a trivialization [CJM02]. Additionally, axiom (SM1) for stable morphisms is here equivalent to

$$\pi^*\varrho = C - \mathrm{curv}(T). \tag{54}$$

The existence of a 2-form ϱ with this property usually has to be derived by descent theory, but here just follows from the axioms. Note that from (54) it follows that $\mathrm{curv}(\mathcal{G}) = \mathrm{d}\varrho$, which shows that the curvature of a topological trivial bundle gerbe is an exact form.

Now I show that bundle gerbe modules, which arise in particular as boundary conditions of open world sheets, are special stable morphisms.

Definition 1.4b. *A bundle gerbe module of a bundle gerbe $\mathcal{G}$ is a 2-form $\omega \in \Omega^2(M)$ together with a stable morphism $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$. The 2-form ω is called the curvature of the bundle gerbe module.*

I write out this definition in detail. Let again $\pi : Y \to M$ be the covering of the gerbe, $L \to Y^{[2]}$ its line bundle with isomorphism μ, and $C \in \Omega^2(Y)$ the curving of $\mathcal{G}$. Assume that the covering of the stable morphism $\mathcal{E}$ is chosen canonically, $Z = Y \times_M M \cong Y$. Then, a bundle gerbe module consists of a vector bundle E over Y, and of an isomorphism

$$\varrho \;:\; L \otimes \pi_2^*E \;\longrightarrow\; \pi_1^*E \tag{55}$$

of vector bundles over $Y^{[2]}$. which is compatible with the isomorphism μ by axiom (SM2) for stable morphisms. This compatibility condition here simplifies to a diagram of isomorphisms of vector bundles over $Z^{[3]} = Y^{[3]}$, namely

$$\begin{array}{ccc}
\pi_{12}^*L \otimes \pi_{23}^*(L \otimes \pi_2^*E) & \xrightarrow{1\otimes\pi_{23}^*\varrho} & \pi_{12}^*L \otimes \pi_{23}^*\pi_1^*E \\
\cong \| & & \| \cong \\
\pi_{12}^*L \otimes \pi_{23}^*L \otimes \pi_3^*E & & \pi_{12}^*(L \otimes \pi_2^*E) \\
\downarrow {\scriptstyle \mu\otimes 1} & & \downarrow {\scriptstyle \pi_{12}^*\varrho} \\
\pi_{13}^*L \otimes \pi_3^*E & & \pi_{12}^*\pi_1^*E \\
\cong \| & & \| \cong \\
\pi_{13}^*(L \otimes \pi_2^*E) & \xrightarrow[\pi_{13}^*\varrho]{} & \pi_{13}^*\pi_1^*E
\end{array} \tag{56}$$

This diagram explains the terminology "module", since it looks like the condition of a (left) action of L on E. Axiom (SM1) here implies

$$\mathrm{curv}(E) = \pi^*\omega - C. \tag{57}$$

From this equation one obtains for the curvature $H = \mathrm{curv}(\mathcal{G})$ of the bundle gerbe $\mathcal{G}$ the property

$$H = \mathrm{d}\omega. \tag{58}$$

In fact Definition 1.4B only covers so-called symmetric bundle gerbe modules, which are important for the applications of bundle gerbes in the Wess-Zumino-Witten-model [Gaw05]. In the mathematical literature [BCM+02, CJM02], one also finds the more general definition of a "non-symmetric" bundle gerbe module. It coincides (apart from the more general coverings here) by dropping axiom (SM1), which restricts the curvature of the vector bundle E. But also from the 2-categorial point of view, the restriction of the curvature comes naturally, because when dropping axiom (SM1) for stable morphisms, Theorem 2.5A wouldn't hold anymore.

In section 2.6 I show how trivializations and bundle gerbe modules are involved in the computation of holonomies of bundle gerbes along open and closed world sheets.

1.5 Morphisms of stable Morphisms

Suppose there are two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over M, with two stable morphisms $\mathcal{A}, \mathcal{A}' : \mathcal{G}_1 \to \mathcal{G}_2$. Recall that the stable morphisms provide vector bundles A and A' over coverings Z and Z' respectively.

DEFINITION 1.5A. *A morphism of stable morphisms*

$$\beta : \mathcal{A} \Longrightarrow \mathcal{A}' \tag{59}$$

consists of

- *a surjective submersion $w : W \to M$ called the covering of β,*
- *two maps $z : W \to Z$ and $z' : W \to Z'$ such that the two diagrams*

$$\begin{array}{ccc} W & \xrightarrow{z'} & Z' \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle y_i'} \\ Z & \xrightarrow[y_i]{} & Y_i \end{array} \tag{60}$$

  commute for $i = 1, 2$, and
- *an isomorphism $\beta : z^*A \to z'^*A'$ of vector bundles over W,*

such that the following axiom (MSM) is satisfied: β is compatible with the isomorphisms α and α' in the sense that the diagram

$$\begin{array}{ccc} z^*y_1^*L_1 \otimes \zeta_2^* z^*A & \xrightarrow{z^*\alpha} & \zeta_1^* z^*A \otimes z^*y_2^*L_2 \\ {\scriptstyle 1\otimes\zeta_2^*\beta}\downarrow & & \downarrow{\scriptstyle \zeta_1^*\beta\otimes 1} \\ z'^*y_1'^*L_1 \otimes \zeta_2^* z'^*A' & \xrightarrow{z'^*\alpha'} & \zeta_1^* z'^*A' \otimes z'^*y_2'^*L_2 \end{array} \tag{61}$$

of isomorphisms of vector bundles over $W^{[2]}$ commutes.

In the diagram I have dropped some occurrences of canonical natural equivalences to improve the readability. I have three remarks on this definition:

(1) It is clear that morphisms $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ of stable morphisms only exist, if the vector bundles A and A' have the same rank.

(2) There is always a canonical choice for the covering W, namely $W := Z \times_{(Y_1 \times_M Y_2)} Z'$. In the case that $Z = Z' = Y_1 \times_M Y_2$ are also chosen canonically, this reduces to $W = Y_1 \times_M Y_2$.

(3) For the canonical choices $W = Z = Z' = Y_1 \times_M Y_2$, and stable isomorphisms $\mathcal{A}$ and $\mathcal{A}'$, Definition 1.5a coincides with the definition of so-called transformations in [Ste00] and with the definition of morphisms of stable isomorphisms in [SSW05].

To compare morphisms of stable morphisms with different coverings, I introduce an equivalence relation on the set of morphisms of stable morphisms.

Definition 1.5b. *Two morphisms $\beta_1 : \mathcal{A} \Rightarrow \mathcal{A}'$ and $\beta_2 : \mathcal{A} \Rightarrow \mathcal{A}'$ of stable morphisms with coverings W_1 and W_2 respectively are considered to be equivalent, if there is a smooth manifold W with surjective submersions $w_i : W \to W_i$ for $i = 1, 2$ such that the diagrams*

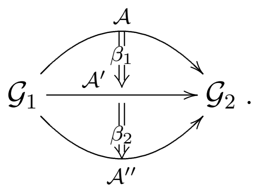
$$\begin{array}{ccc} W & \xrightarrow{w_2} & W_2 \\ {\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle z_2} \\ W_1 & \xrightarrow[z_1]{} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W & \xrightarrow{w_2} & W_2 \\ {\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle z_2'} \\ W_1 & \xrightarrow[z_1']{} & Z' \end{array} \tag{62}$$

commute, and the isomorphisms β_1 and β_2 coincide when pulled back to W, i.e.

$$w_1^*\beta_1 = w_2^*\beta_2. \tag{63}$$

Note that equation (63) makes sense because of the commutativity of the diagrams (62). It is easy to verify that Definition 1.5b defines an equivalence relation.

If there is a third stable morphism $\mathcal{A}'' : \mathcal{G}_1 \to \mathcal{G}_2$, and two morphisms $\beta_1 : \mathcal{A} \Rightarrow \mathcal{A}'$ and $\beta_2 : \mathcal{A}' \Rightarrow \mathcal{A}''$ of stable morphisms, both morphisms can be composed. Diagrammatically, the composition of morphisms of stable morphisms is denoted by



(64)

Definition 1.5c. *The composition of the morphisms $\beta_1 : \mathcal{A} \Rightarrow \mathcal{A}'$ and $\beta_2 : \mathcal{A}' \Rightarrow \mathcal{A}''$ is the morphism*

$$\beta_2 \circ \beta_1 \; : \; \mathcal{A} \Longrightarrow \mathcal{A}'' \tag{65}$$

whose covering is the fibre product $W := W_1 \times_{Z'} W_2$ in the commuting diagram

$$\begin{array}{ccc} W & \xrightarrow{w_2} & W_2 \\ {\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle z_2'} \\ W_1 & \xrightarrow[z_1']{} & Z' \end{array} \tag{66}$$

together with the projections $z := z_1 \circ w_1$ *and* $z'' := z''_2 \circ w_2$ *to the coverings* Z *and* Z'' *of* $\mathcal{A}$ *and* $\mathcal{A}''$ *respectively, and whose isomorphism is*

$$w_2^*\beta_2 \circ w_1^*\beta_1 \; : \; z^*A \longrightarrow z''^*A''. \tag{67}$$

The composed morphism $\beta_2 \circ \beta_1$ clearly satisfies the axiom (MSM) for morphisms of stable morphisms. It is furthermore well-defined on equivalence classes of morphisms of stable morphisms: if one takes other representatives β'_1 and β'_2 with coverings W'_1 and W'_2 respectively, so that there exist spaces $\hat{W}_1$ and $\hat{W}_2$ according to Definition 1.5b, then the compositions $\beta_2 \circ \beta_1$ and $\beta'_2 \circ \beta'_1$ coincide when pulled back to $\hat{W} := \hat{W}_1 \times_{Z'} \hat{W}_2$.

2 The 2-Category of Bundle Gerbes

The structure defined in the previous chapter fits in the framework of a 2-category. The aim of the present chapter is to make this statement precise: I define the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over a smooth manifold M.

In section 2.1 I give the definition of a 2-category, consisting of structure and axioms. In the following two sections, I relate the definitions of the first chapter to the structure of the 2-category of bundle gerbes. In section 2.4 I check that the axioms of a 2-category are satisfied by this structure. This completes the definition of $\mathfrak{BGrb}(M)$.

In section 2.5 I relate the definition of a stable isomorphism from section 1.3 to the notion of a 1-isomorphism one has in every 2-category, and show that they coincide. This is applied in section 2.6 to give short proofs of the well-definedness of the definitions of holonomy along world sheets with and without boundaries.

2.1 The Concept of a 2-Category

The definition of a 2-category given below is the straight-forward categorification of Definition 1.1A of a category. The main concept is to replace equalities by natural equivalences which satisfy certain natural coherence conditions. So, the three equalities in the axioms (C1) and (C2) are replaced by three natural equivalences. This requires that the composition map is replaced by a composition functor, and the identity morphism is replaced by an identity functor. In turn, this requires that the sets of morphisms become categories. This way one has arrived at the following definition.

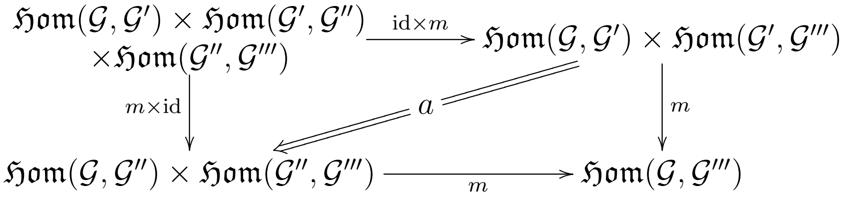
DEFINITION 2.1A. *A 2-category $\mathfrak{C}$ consists of the following data:*

- *A set of objects* $\mathrm{Obj}(\mathfrak{C})$.
- *For each pair $\mathcal{G},\mathcal{G}'$ of objects a category $\mathfrak{Hom}(\mathcal{G},\mathcal{G}')$.*
- *For each triple $\mathcal{G},\mathcal{G}',\mathcal{G}''$ of objects a functor*

$$m : \mathfrak{Hom}(\mathcal{G},\mathcal{G}') \times \mathfrak{Hom}(\mathcal{G}',\mathcal{G}'') \longrightarrow \mathfrak{Hom}(\mathcal{G},\mathcal{G}''), \qquad (68)$$

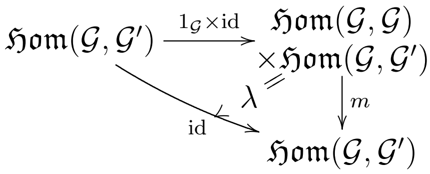
called the composition functor,

together with a natural equivalence of functors

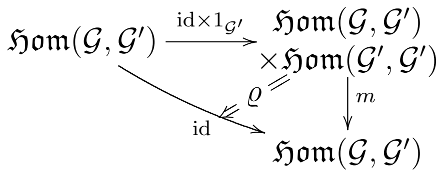


for each quadruple $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$, $\mathcal{G}'''$ of objects.

- *For each object $\mathcal{G}$ a functor $1_\mathcal{G} : \mathbb{1} \to \mathfrak{Hom}(\mathcal{G},\mathcal{G})$ together with natural equivalences*



and



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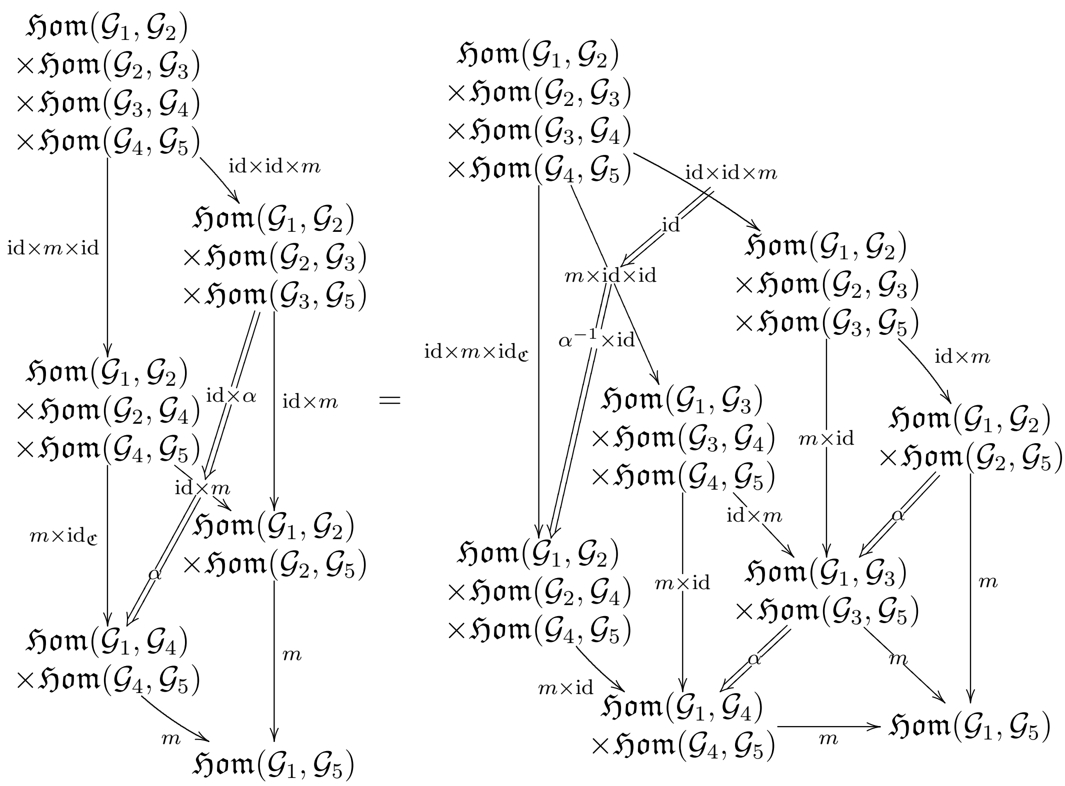
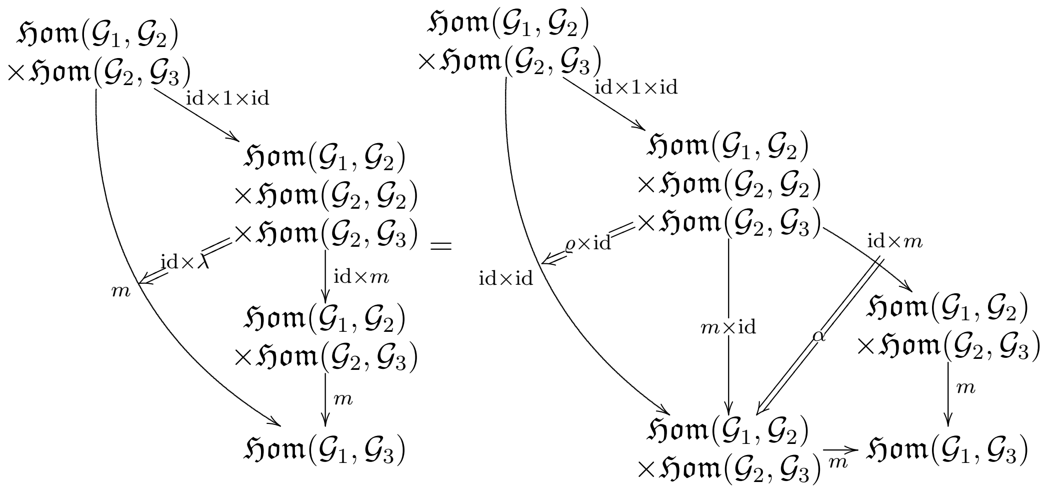
for each two objects $\mathcal{G}$ and $\mathcal{G}'$.

This structure has to satisfy two coherence axioms:

(2C1) The Pentagon identity:



(2C2) The compatibility of λ and ϱ with α:



As a trivial example of a 2-category, take any monoidal category $(\mathfrak{C}, \otimes, 1, \alpha, \lambda, \varrho)$, and define a 2-category with only one object $\bullet$ by $\mathfrak{Hom}(\bullet, \bullet) := \mathfrak{C}$, $m := \otimes$ and $1_\bullet = 1$. Then the rest of the data and axioms reduce to the data and axioms of a monoidal category.

For two objects $\mathcal{G}$, $\mathcal{G}'$ of a 2-category $\mathfrak{C}$, a 1-morphism $\mathcal{A}$ is an object of the category $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}')$ and is denoted by $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$. A 2-morphism is a morphism $\beta : \mathcal{A}_1 \to \mathcal{A}_2$ in the category $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}')$ and – as a 2-morphism of $\mathfrak{C}$ – is denoted by $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$. This notation comes from the 2-category $\mathfrak{Cat}$ of categories, whose 1-morphisms are functors, and whose 2-morphisms are natural transformations.

A 2-morphism $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ of 1-morphisms $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{G} \to \mathcal{G}'$ is called a 2-isomorphism, if it is an isomorphism in the category $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}')$; equivalently: if there is another 2-morphism $\beta' : \mathcal{A}_2 \Rightarrow \mathcal{A}_1$ such that $\beta' \circ \beta = \mathrm{id}_{\mathcal{A}_1}$ and $\beta \circ \beta' = \mathrm{id}_{\mathcal{A}_2}$. A 1-morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ is called 1-isomorphism, if there is another 1-morphism $\mathcal{A}' : \mathcal{G}' \to \mathcal{G}$, and two 2-isomorphisms $i_r : m(\mathcal{A}, \mathcal{A}') \Rightarrow 1_{\mathcal{G}}(\bullet)$ and $i_l : m(\mathcal{A}', \mathcal{A}) \Rightarrow 1_{\mathcal{G}'}(\bullet)$. It can be shown that the "inverse"1-morphism $\mathcal{A}'$ is unique up to isomorphisms in $\mathfrak{Hom}(\mathcal{G}', \mathcal{G})$, i.e. unique up to 2-isomorphisms in $\mathfrak{C}$. The 1-isomorphisms form, together with all 2-morphism between them, a full subcategory $\mathfrak{Iso}(\mathcal{G}, \mathcal{G}')$ of $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}')$.

Sometimes (small) 2-categories are defined as a set of objects, a set of 1-morphisms and set of 2-morphisms, together with one composition for 1-morphisms, and two compositions for 2-morphisms, called horizontal and vertical composition, and with various axioms. This is equivalent to Definition 2.1A: the composition of 1-morphisms is the functor m on objects of $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}') \times \mathfrak{Hom}(\mathcal{G}', \mathcal{G}'')$, and the two compositions of 2-morphisms are given by the functor m acting on morphisms of $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}') \times \mathfrak{Hom}(\mathcal{G}', \mathcal{G}'')$ (horizontal composition, cf. (71)) and by the composition map of the category $\mathfrak{Hom}(\mathcal{G}, \mathcal{G}')$ (vertical composition, cf. (64)). Then the axioms (2C1) and (2C2) are stated as compatibility conditions between horizontal and vertical compositions.

2.2 Structure I: Objects and Functors

To define the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over a smooth manifold M, I first have to specify the structure. In this section, the objects $\mathcal{G}$, the morphism categories $\mathfrak{Hom}(\mathcal{G},\mathcal{G}')$, the composition functor m and the functor $1_{\mathcal{G}}$ are defined.

The objects are of course bundle gerbes over M.

For each two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over M define the category $\mathfrak{Hom}(\mathcal{G}_1,\mathcal{G}_2)$ as follows. Its objects are stable morphisms $\mathcal{A}:\mathcal{G}_1\to\mathcal{G}_2$, its morphisms are equivalence classes of morphisms $\beta:\mathcal{A}\Rightarrow\mathcal{A}'$ of stable morphisms under the equivalence relation from Definition 1.5B, and the composition is the composition of morphisms of stable morphisms as defined in section 1.5. For every stable morphism $\mathcal{A}$ there is the identity morphism $\mathrm{id}:\mathcal{A}\Rightarrow\mathcal{A}$, which has the same covering as $\mathcal{A}$, and the identity isomorphism. Both axioms (C1) and (C2) of a category are satisfied. Additionally, each morphism β is invertible, hence $\mathfrak{Hom}(\mathcal{G}_1,\mathcal{G}_2)$ is a groupoid, and I call it the groupoid of stable morphisms.

In particular, if $\mathcal{G}$ is a gerbe over M, $\mathfrak{End}(\mathcal{G}):=\mathfrak{Hom}(\mathcal{G},\mathcal{G})$ is called the groupoid of gauge transformations of $\mathcal{G}$, and $\mathfrak{Aut}(\mathcal{G}):=\mathfrak{Iso}(\mathcal{G},\mathcal{G})$ is called the groupoid of gauge equivalences of $\mathcal{G}$. Such categories have been postulated by N. Hitchin in his lecture at the Arbeitstagung 2001 of the MPIM in Bonn.

So the 1-morphisms of $\mathfrak{BGrb}(M)$ are stable morphisms, and the 2-morphisms are morphisms of stable morphisms, and every 2-morphism is even a 2-isomorphism.

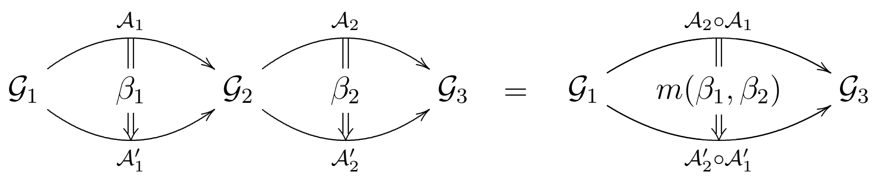
For three bundle gerbes $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ I define the composition functor

$$m\;:\;\mathfrak{Hom}(\mathcal{G}_1,\mathcal{G}_2)\times\mathfrak{Hom}(\mathcal{G}_2,\mathcal{G}_3)\;\longrightarrow\;\mathfrak{Hom}(\mathcal{G}_1,\mathcal{G}_3)\tag{69}$$

as follows. It is quite clear that on objects $\mathcal{A}_1:\mathcal{G}_1\to\mathcal{G}_2$ and $\mathcal{A}_2:\mathcal{G}_2\to\mathcal{G}_3$ it is defined by the composition of stable morphisms from Definition 1.3C,

$$m(\mathcal{A}_1,\mathcal{A}_2):=\mathcal{A}_2\circ\mathcal{A}_1.\tag{70}$$

Now let $\beta_1:\mathcal{A}_1\Rightarrow\mathcal{A}_1'$ and $\beta_2:\mathcal{A}_2\Rightarrow\mathcal{A}_2'$ be two morphisms of stable morphisms representing morphisms in the categories $\mathfrak{Hom}(\mathcal{G}_1,\mathcal{G}_2)$ and $\mathfrak{Hom}(\mathcal{G}_2,\mathcal{G}_3)$ respectively. The composition I have to define is graphically denoted by the following diagram:



(71)

Recall that the compositions $\mathcal{A}_2\circ\mathcal{A}_1$ and $\mathcal{A}_2'\circ\mathcal{A}_1'$ of stable morphisms consist of vector bundles $\tilde{A}:=z_1^*A_1\otimes z_2^*A_2$ over $\tilde{Z}:=Z_1\times_{Y_2}Z_2$ and $\tilde{A}':=z_1'^*{A_1'}^*\otimes z_2'^*A_2'$ over $\tilde{Z}':=Z_1'\times_{Y_2}Z_2'$. Recall further that the morphisms β_1 and β_2 of stable morphisms are isomorphisms $\beta_i:z_i^*A_i\to z_i'^*A_i'$ of vector bundles over spaces W_i for $i=1,2$. Now choose the covering

$$W:=W_1\times_{Y_2}W_2\tag{72}$$

for the morphism $m(\beta_1,\beta_2)$. Here W_i projects to Y_2 via $y_2\circ z_i=y_2'\circ z_i'$, where the equality comes from the commuting diagram (60) in the definition of a morphism of stable morphisms. I also have to define projections $\tilde{z} : W \to \tilde{Z}$ and $\tilde{z}' : W \to \tilde{Z}'$, which is done by sending an element $(w_1, w_2) \in W$ to $(z_1(w_1), z_2(w_2)) \in Z_1 \times Z_2$, which in fact lies in $\tilde{Z}$ since the fibre product (72) is taken over Y_2, and analogous for the primed quantities. Also by the properties of the coverings W_1 and W_2 one obtains the commuting diagrams

$$\begin{array}{ccc} W & \xrightarrow{\tilde{z}'} & \tilde{Z}' \\ {\scriptstyle \tilde{z}}\downarrow & & \downarrow{\scriptstyle y_i'} \\ \tilde{Z} & \xrightarrow[y_i]{} & Y_i \end{array} \tag{73}$$

for $i = 1, 3$. So, $W$is a valid choice for the covering of $m(\beta_1, \beta_2)$. By construction, we have the relations

$$z_i \circ \tilde{z} = z_i \circ w_i \quad \text{and} \quad z_i' \circ \tilde{z}' = z_i' \circ w_i \tag{74}$$

as maps from W to Z_i and $\tilde{Z}_i$ respectively. Now I am in the position to define the isomorphism $\hat{\beta}$ of the morphism $m(\beta_1, \beta_2)$ by

$$\hat{\beta} := w_1^*\beta_1 \otimes w_2^*\beta_2 \ : \ \tilde{z}^*\tilde{A} \ \longrightarrow \ \tilde{z}'^*\tilde{A}', \tag{75}$$

which is well-defined due to the relations (74) and the definitions of β_i, $\tilde{A}$ and $\tilde{A}'$. One can easily check that the axiom (MSM) for morphisms of stable isomorphisms is satisfied, since I just took the pullback and the tensor product of β_1 and β_2, which satisfy the axiom. It is also easy to see that the definition of $m(\beta_1, \beta_2)$ does not depend on the choice of representatives β_1 and β_2.

One may check immediately, that by (70) and (75) a functor is defined: the composition of morphisms of stable morphisms is respected, and the identity morphisms of two stable morphisms are mapped to the identity morphism of the composition of the stable morphisms.

Consider a single bundle gerbe $\mathcal{G}$ over M. To define the functor $1_{\mathcal{G}} : \mathbb{1} \to \mathfrak{Hom}(\mathcal{G}, \mathcal{G})$, it is sufficient to define the image $1_{\mathcal{G}}(\bullet)$ of the object $\bullet$ in $\mathbb{1}$. Let $L \to Y^{[2]}$ be the line bundle of the bundle gerbe $\mathcal{G}$ and μ its isomorphism. For the definition of the stable isomorphism $\mathcal{L}_{\mathcal{G}} := 1_{\mathcal{G}}(\bullet)$ choose the covering $Z = Y^{[2]}$ and the line bundle L itself. Then note that

$$l := \pi_{124}^*\mu^{-1} \circ \pi_{134}^*\mu \ : \ \pi_{13}^*L \otimes \pi_{34}^*L \ \longrightarrow \ \pi_{12}^*L \otimes \pi_{24}^*L \tag{76}$$

is an isomorphism of line bundles over $Z^{[2]} = Y^{[4]}$. So define $\mathcal{L}_{\mathcal{G}} := (L, l)$, which gives in fact a stable isomorphism, since L is a line bundle and the gerbe axiom (G1) for the curvature of L is here equivalent to the axiom (SM1) for the curvature of the line bundle of a stable isomorphism. Axiom (SM2) is equivalent to three copies of axiom (G2) for the bundle gerbe $\mathcal{G}$.

2.3 Structure II: Natural Equivalences

For a 1-morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ the natural equivalences λ and ϱ are 2-morphisms

$$\lambda_{\mathcal{A}} : m(1_{\mathcal{G}}(\bullet), \mathcal{A}) \quad \Longrightarrow \quad \mathcal{A} \tag{77}$$

$$\varrho_{\mathcal{A}} : m(\mathcal{A}, 1_{\mathcal{G}'}(\bullet)) \quad \Longrightarrow \quad \mathcal{A} \tag{78}$$

and the condition (4) on natural transformations is, that for every 2-morphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ the diagrams

$$\begin{array}{ccc} m(1_{\mathcal{G}}(\bullet), \mathcal{A}) & \xRightarrow{\lambda_{\mathcal{A}}} & \mathcal{A} \\ {\scriptstyle m(\mathrm{id},\beta)}\Downarrow & & \Downarrow{\scriptstyle \beta} \\ m(1_{\mathcal{G}}(\bullet), \mathcal{A}') & \xRightarrow[\lambda_{\mathcal{A}'}]{} & \mathcal{A}' \end{array} \quad \text{and} \quad \begin{array}{ccc} m(\mathcal{A}, 1_{\mathcal{G}'}(\bullet)) & \xRightarrow{\varrho_{\mathcal{A}}} & \mathcal{A} \\ {\scriptstyle m(\beta,\mathrm{id})}\Downarrow & & \Downarrow{\scriptstyle \beta} \\ m(\mathcal{A}', 1_{\mathcal{G}'}(\bullet)) & \xRightarrow[\varrho_{\mathcal{A}'}]{} & \mathcal{A}' \end{array} \tag{79}$$

of 2-morphisms commute.

In the case of the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M one is concerned with the composition of the identity stable isomorphism $1_{\mathcal{G}}(\bullet) = \mathcal{L}_{\mathcal{G}}$ defined in section 2.2, with a stable morphism $\mathcal{A} = (A, \alpha)$. Recall that A is a vector bundle over some covering Z and that the composition is the stable morphism $\mathcal{A} \circ \mathcal{L}_{\mathcal{G}}$ with covering $\tilde{Z} = Y^{[2]} \times_Y Z$, vector bundle $\tilde{A} = y^*L \otimes z^*A$ and isomorphism $\tilde{\alpha} = (\mathrm{id} \otimes z^*\alpha) \circ (y^*l \otimes \mathrm{id})$. For the definition of the morphism $\lambda_{\mathcal{A}} : \mathcal{A} \circ \mathcal{L}_{\mathcal{G}} \Rightarrow \mathcal{A}$ take the covering Z with the map $\mathrm{id} : Z \to Z$ to the covering of $\mathcal{A}$ and the map

$$\tilde{z} \;:\; Z \longrightarrow \tilde{Z} \;:\; z \longmapsto (y(z), y(z), z) \tag{80}$$

to the covering of $\mathcal{A} \circ \mathcal{L}_{\mathcal{G}}$, which is a section of the projection $z : \tilde{Z} \to Z$. Note that the two diagrams (60) from the definition of a morphism of stable morphisms commute, so Z is a valid covering for $\lambda_{\mathcal{A}}$. Also the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{z}} & \tilde{Z} \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle y} \\ Y & \xrightarrow[\Delta]{} & Y^{[2]} \end{array} \tag{81}$$

commutes, so that one obtains $\tilde{z}^*\tilde{A} = y^*\Delta^*L \otimes A$. Now I define

$$\lambda_{\mathcal{A}} := \lambda_A \circ (y^*t_\mu \otimes 1) \;:\; \tilde{z}^*\tilde{A} \;\longrightarrow\; A, \tag{82}$$

where $\lambda_A : 1 \otimes A \to A$ is the canonical natural equivalence of the monoidal category $\mathfrak{Bun}(Z)$.

Proposition 2.3a. *The isomorphism $\lambda_{\mathcal{A}}$ from equation (82) defines a morphism of stable morphisms $\lambda_{\mathcal{A}} : \mathcal{A} \circ \mathcal{L}_{\mathcal{G}} \Rightarrow \mathcal{A}$, and the left diagram of (79) commutes.*

Proof . First I show that $\lambda_{\mathcal{A}}$ satisfies axiom (MSM), which demands that the diagram

$$\begin{array}{ccc} y^*L \otimes \zeta_2^*\tilde{z}^*\tilde{A} & \xrightarrow{\tilde{z}^*\tilde{\alpha}} & \zeta_1^*\tilde{z}^*\tilde{A} \otimes y'^*L' \\ {\scriptstyle 1\otimes\zeta_2^*\lambda_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \zeta_1^*\lambda_{\mathcal{A}}\otimes 1} \\ y^*L \otimes \zeta_2^*A & \xrightarrow{\alpha} & \zeta_1^*A \otimes y'^*L' \end{array} \tag{83}$$

of isomorphisms of vector bundles over $Z^{[2]}$ commutes. Pasting in the definition of $\tilde{\alpha}$, of the isomorphism l and of $\lambda_{\mathcal{A}}$, I obtain the diagram

$$\begin{array}{ccc} * & \xrightarrow{1\otimes\alpha} & * \\ {\scriptstyle \tilde{z}^*y^*\mu_{124}^*\mu\otimes 1}\downarrow & & \\ * & & \\ {\scriptstyle \tilde{z}^*y^*\pi_{134}^*\mu^{-1}\otimes 1}\downarrow & & \downarrow{\scriptstyle \lambda_A(\zeta_1^*y^*t_\mu\otimes 1)\otimes 1} \\ * & & \\ {\scriptstyle 1\otimes\lambda_A(\zeta_2^*y^*t_\mu\otimes 1)}\downarrow & & \\ * & \xrightarrow{\alpha} & * \end{array} \tag{84}$$

Note that by axiom (MC2) of the monoidal category $\mathfrak{Bun}(Z)$ one has $1\otimes\lambda_A(\zeta_2^*y^*t_\mu\otimes 1) = \varrho_A(1\otimes\zeta_2^*y^*t_\mu)\otimes 1$. Now consider the pullback of the associativity axiom (G2) for the isomorphism μ of the bundle gerbe $\mathcal{G}$ by $\Delta_{1122} : Y^{[2]} \to Y^{[4]}$, which is

$$\Delta_{122}^*\mu \circ (\Delta_{112}^*\mu \otimes 1) = \Delta_{112}^*\mu \circ (1 \otimes \Delta_{122}^*\mu). \tag{85}$$

Observe that by construction of the maps y and $\tilde{z}$ we also have the relations

$$\pi_{134} \circ y \circ \tilde{z} = \Delta_{122}^* \circ y \quad \text{and} \quad \pi_{124} \circ y \circ \tilde{z} = \Delta_{112} \circ y. \tag{86}$$

Combining (85) and (86) with Proposition 1.2B one obtains that the three isomorphisms on the left hand side of the diagram (84) compose to $\lambda_A(y^*\pi_1^*t_\mu\otimes 1)\otimes 1 = \lambda_A(\zeta_1^*y^*t_\mu \otimes 1)\otimes 1$, so that the diagram obviously commutes.

For the second part of the proof I rewrite the left diagram of (79) in terms of isomorphisms of line bundles. Let W be the covering of the morphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ with maps $z : W \to Z$ and $z' : W \to Z'$. By definition of the composition of morphisms, $\beta \circ \lambda_{\mathcal{A}}$ has the covering $Z \times_Z W \cong W$, and $\lambda_{\mathcal{A}'} \circ m(\mathrm{id}, \beta)$ has the covering $(Y^{[2]} \times_Y W) \times_{(Y^{[2]}\times_Y Z')} Z' \cong W$. So the two morphisms of stable morphisms I have to compare already are defined over the same covering, hence the diagram is equivalent to a diagram of isomorphisms of line bundles over W, namely

$$\begin{array}{ccc} z^*\tilde{z}^*\tilde{A} & \xrightarrow{t_\mu\otimes 1} & z^*A \\ {\scriptstyle 1\otimes\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ z'^*\tilde{z}'^*\tilde{A}' & \xrightarrow[t_\mu\otimes 1]{} & z'^*A' \end{array} \tag{87}$$

which obviously commutes. □

The discussion of the natural equivalence ϱ is completely analogous to the one of λ, here one ends up with a morphism $\varrho_{\mathcal{A}} : \mathcal{L}_{\mathcal{G}'} \circ \mathcal{A} \Rightarrow \mathcal{A}$ of stable morphisms, which is defined by

$$\varrho_{\mathcal{A}} := \varrho_A \circ (1 \otimes y'^*t_{\mu'}). \tag{88}$$

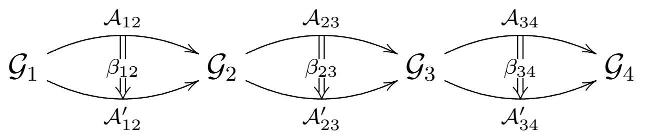
Now I come to the definition of the natural equivalence α. For three 1-morphisms

$$\mathcal{G}_1 \xrightarrow{\mathcal{A}_{12}} \mathcal{G}_2 \xrightarrow{\mathcal{A}_{23}} \mathcal{G}_3 \xrightarrow{\mathcal{A}_{34}} \mathcal{G}_4 \tag{89}$$

the natural equivalence α is a 2-isomorphism

$$\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}} \;:\; m(\mathcal{A}_{12}, m(\mathcal{A}_{23},\mathcal{A}_{34})) \Longrightarrow m(m(\mathcal{A}_{12},\mathcal{A}_{23}),\mathcal{A}_{34}), \tag{90}$$

and the condition (4) on natural transformations is, that for three 2-morphisms



(91)

the diagram

$$\begin{array}{ccc} m(\mathcal{A}_{12}, m(\mathcal{A}_{23},\mathcal{A}_{34})) & \xRightarrow{\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}}} & m(m(\mathcal{A}_{12},\mathcal{A}_{23}),\mathcal{A}_{34}) \\ {\scriptstyle m(\beta_{12},m(\beta_{23},\beta_{34}))}\Big\Downarrow & & \Big\Downarrow{\scriptstyle m(m(\beta_{12},\beta_{23}),\beta_{34})} \\ m(\mathcal{A}'_{12}, m(\mathcal{A}'_{23},\mathcal{A}'_{34})) & \xRightarrow{\alpha_{\mathcal{A}'_{12},\mathcal{A}'_{23},\mathcal{A}'_{34}}} & m(m(\mathcal{A}'_{12},\mathcal{A}'_{23}),\mathcal{A}'_{34}) \end{array} \tag{92}$$

of 2-morphisms commutes.

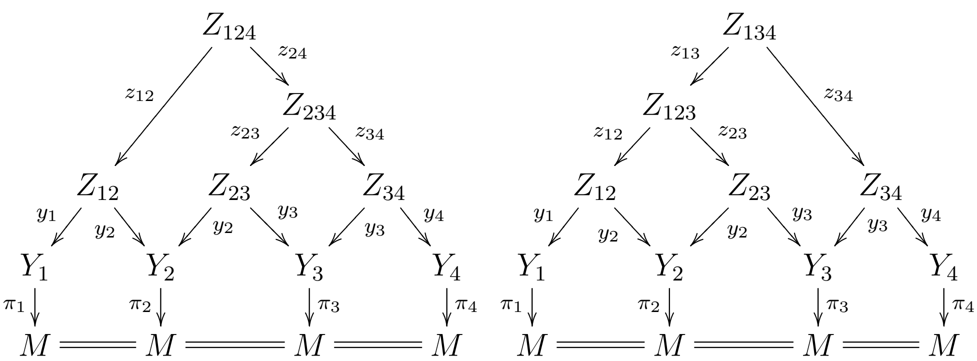
In the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M one is concerned with the two ways

$$m(\mathcal{A}_{12}, m(\mathcal{A}_{23},\mathcal{A}_{34})) = (\mathcal{A}_{34}\circ\mathcal{A}_{23})\circ\mathcal{A}_{12} \;:=\; (A_{1(24)},\alpha_{1(24)}) \tag{93}$$
$$m(m(\mathcal{A}_{12},\mathcal{A}_{23}),\mathcal{A}_{34}) = \mathcal{A}_{34}\circ(\mathcal{A}_{23}\circ\mathcal{A}_{12}) \;:=\; (A_{(13)4},\alpha_{(13)4}) \tag{94}$$

of composing stable morphisms. The natural equivalence α here is a morphism

$$\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}} \;:\; (A_{1(24)},\alpha_{1(24)}) \Longrightarrow (A_{(13)4},\alpha_{(13)4}) \tag{95}$$

of stable morphisms. The following diagrams correspond to diagram (39) from section 1.3, applied to the two ways:



(96)

In the left diagram, first the composition $\mathcal{A}_{24} := \mathcal{A}_{34}\circ\mathcal{A}_{23}$ is computed, while in the diagram on the right hand side first the composition $\mathcal{A}_{13} := \mathcal{A}_{23}\circ\mathcal{A}_{12}$ is done. By definition of the composition of stable morphisms, the spaces Z_{123}, Z_{234}, Z_{134} and Z_{124} are fibre products, in particular there is a canonical diffeomorphism

$$Z_{134} = (Z_{12}\times_{Y_2} Z_{23})\times_{Y_3} Z_{34} \longrightarrow Z_{12}\times_{Y_2}(Z_{23}\times_{Y_3} Z_{34}) = Z_{124}, \tag{97}$$

by which I identify the two spaces and call them just W in the following. This allows me to choose W to be the covering of the morphism $\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}}$ with the identity maps to Z_{134} and Z_{124}. Recall further that the vector bundles of the composed stable morphisms are

$$A_{1(24)} = z_{12}^*A_{12} \otimes z_{24}^*(z_{23}^*A_{23} \otimes z_{34}^*A_{34}) \tag{98}$$
$$A_{(13)4} = z_{13}^*(z_{12}^*A_{12} \otimes z_{23}^*A_{23}) \otimes z_{34}^*A_{34}. \tag{99}$$

Recall from Definition 1.1B that the monoidal category $\mathfrak{Bun}(W)$ comes with a natural equivalence α, which is an isomorphism

$$\alpha_{B_1,B_2,B_3} : B_1 \otimes (B_2 \otimes B_3) \longrightarrow (B_1 \otimes B_2) \otimes B_3 \tag{100}$$

for any three vector bundles B_1, B_2 and B_3 over W. So define $\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}}$ to be the isomorphism

$$\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}} := \alpha_{z_{12}^*A_{12},z_{23}^*A_{23},z_{34}^*A_{34}} : A_{1(24)} \longrightarrow A_{(13)4}. \tag{101}$$

I have to show that the axiom (MSM) for morphisms of stable morphisms is satisfied. Here this is the commutativity of the diagram

$$\begin{array}{ccc} y_1^*L_1 \otimes \zeta_2^*A_{1(24)} & \xrightarrow{\alpha_{1(24)}} & \zeta_1^*A_{1(24)} \otimes y_4^*L_4 \\ {\scriptstyle 1\otimes\zeta_2^*\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}}}\downarrow & & \downarrow{\scriptstyle \zeta_1^*\alpha_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}}\otimes 1} \\ y_1^*L_1 \otimes \zeta_2^*A_{(13)4} & \xrightarrow[\alpha_{(13)4}]{} & \zeta_1^*A_{(13)4} \otimes y_4^*L_4 \end{array} \tag{102}$$

of vector bundles over $W^{[2]}$. Recall that the isomorphisms of the composed stable morphisms are defined by

$$\begin{aligned} \alpha_{1(24)} = \alpha_{A_{12},A_{23}\otimes A_{34},L_3} &\circ (1 \otimes \alpha_{A_{23},A_{34},L_4}) \circ (1 \otimes 1 \otimes z_{34}^*\alpha_{34}) \\ &\circ (1 \otimes \alpha^{-1}_{A_{23},L_3,A_{34}}) \circ (1 \otimes z_{23}^*\alpha_{23} \otimes 1) \circ (1 \otimes \alpha_{L_2,A_{23},A_{34}}) \\ &\circ \alpha^{-1}_{A_{12},L_2,A_{23}\otimes A_{34}} \circ (z_{12}^*\alpha_{12} \otimes 1) \circ \alpha_{L_1,A_{12},A_{23}\otimes A_{34}} \end{aligned} \tag{103}$$

and

$$\begin{aligned} \alpha_{(13)4} = \alpha_{A_{12}\otimes A_{23},A_{34},L_3} &\circ (1 \otimes z_{34}^*\alpha_{34}) \circ \alpha^{-1}_{A_{12}\otimes A_{23},L_2,A_{34}} \circ (\alpha_{L_1,A_{12},A_{23}} \otimes 1) \\ &\circ (1 \otimes z_{23}^*\alpha_{23} \otimes 1) \circ (\alpha^{-1}_{A_{12},L_2,A_{23}} \otimes 1) \circ (z_{12}^*\alpha_{12} \otimes 1 \otimes 1) \\ &\circ (\alpha_{L_1,A_{12},A_{23}} \otimes 1) \circ \alpha_{L_1,A_{12}\otimes A_{23},A_{34}}. \end{aligned} \tag{104}$$

Up to occurrences of the natural equivalence α these two isomorphisms coincide, and the commutativity of diagram (102) is equivalent to the Pentagon identity (MC1) of the monoidal category $\mathfrak{Bun}(W^{[2]})$.

It remains to show that α defined in (101) is a natural equivalence, i.e. the commutativity of the diagram (92). I rewrite this diagram in terms of isomorphisms of vector bundles over Z, namely

$$\begin{array}{ccc} z_{12}^*A_{12} \otimes z_{24}^*(z_{23}^*A_{23} \otimes z_{34}^*A_{34}) & \xrightarrow{\alpha_{z_{12}^*A_{12},z_{23}^*A_{23},z_{34}^*A_{34}}} & z_{13}^*(z_{12}^*A_{12} \otimes z_{23}^*A_{23}) \otimes z_{34}^*A_{34} \\ {\scriptstyle z_{12}^*\beta_{12}\otimes z_{24}^*(z_{23}^*\beta_{23}\otimes z_{34}^*\beta_{34})}\downarrow & & \downarrow{\scriptstyle z_{13}^*(z_{12}^*\beta_{12}\otimes z_{23}^*\beta_{23})\otimes z_{34}^*\beta_{34}} \\ z_{12}^*A'_{12} \otimes z_{24}^*(z_{23}^*A'_{23} \otimes z_{34}^*A'_{34}) & \xrightarrow{\alpha_{z_{12}^*A'_{12},z_{23}^*A'_{23},z_{34}^*A'_{34}}} & z_{13}^*(z_{12}^*A'_{12} \otimes z_{23}^*A'_{23}) \otimes z_{34}^*A'_{34} \end{array}$$

Then the commutativity of this diagram is exactly the condition (4) that α is a natural transformation.

2.4 Axioms

To complete the definition of the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M, I have to show, that the axioms (2C1) and (2C2) are satisfied. The Pentagon identity (2C1) means for five bundle gerbes $\mathcal{G}_1, ..., \mathcal{G}_5$ and four stable morphisms $\mathcal{A}_{i,i+1}$ for $i = 1, ..., 4$ the commutativity of the following diagram:

$$\begin{array}{ccc}
m(\mathcal{A}_{12}, m(\mathcal{A}_{23}, m(\mathcal{A}_{34}, \mathcal{A}_{45}))) & \xrightarrow{m(\mathrm{id}_{\mathcal{A}_{12}}, \alpha_{\mathcal{A}_{23},\mathcal{A}_{34},\mathcal{A}_{45}})} & m(\mathcal{A}_{12}, m(m(\mathcal{A}_{23}, \mathcal{A}_{34}), \mathcal{A}_{45})) \\
\downarrow \alpha_{\mathcal{A}_{12},\mathcal{A}_{23},m(\mathcal{A}_{34},\mathcal{A}_{45})} & & \\
m(m(\mathcal{A}_{12}, \mathcal{A}_{23}), m(\mathcal{A}_{34}, \mathcal{A}_{45})) & & \downarrow \alpha_{\mathcal{A}_{12},m(\mathcal{A}_{23},\mathcal{A}_{34}),\mathcal{A}_{45}} \\
\downarrow \alpha_{m(\mathcal{A}_{12},\mathcal{A}_{23}),\mathcal{A}_{34},\mathcal{A}_{45}} & & \\
m(m(m(\mathcal{A}_{12}, \mathcal{A}_{23}), \mathcal{A}_{34}), \mathcal{A}_{45}) & \xrightarrow[m(\alpha^{-1}_{\mathcal{A}_{12},\mathcal{A}_{23},\mathcal{A}_{34}}, \mathrm{id}_{\mathcal{A}_{45}})]{} & m(m(\mathcal{A}_{12}, m(\mathcal{A}_{23}, \mathcal{A}_{34})), \mathcal{A}_{45})
\end{array}$$

This form of the diagram from axiom (2C1) makes the name pentagon axiom [ML97] obvious. To check the commutativity of this diagram, I work in the same notation as in section 2.3. The coverings of all five compositions of stable morphisms are again all canonically isomorphic and thus all identified with some space W. Then the diagram is a diagram of isomorphisms of line bundles over W, namely:

$$\begin{array}{ccc}
z_{12}^*A_{12} \otimes (z_{23}^*A_{23} \otimes (z_{34}^*A_{34} \otimes z_{45}^*A_{45})) & \xrightarrow{1\otimes\alpha_{z_{23}^*A_{23},z_{34}^*A_{34},z_{45}^*A_{45}}} & z_{12}^*A_{12} \otimes ((z_{23}^*A_{23} \otimes z_{34}^*A_{34}) \otimes z_{45}^*A_{45}) \\
\downarrow \alpha_{z_{12}^*A_{12},z_{23}^*A_{23},z_{34}^*A_{34}\otimes z_{45}^*A_{45}} & & \\
(z_{12}^*A_{12} \otimes z_{23}^*A_{23}) \otimes (z_{34}^*A_{34} \otimes z_{45}^*A_{45}) & & \downarrow \alpha_{z_{12}^*A_{12},z_{23}^*A_{23}\otimes z_{34}^*A_{34},A_{45}} \\
\downarrow z_{12}^*\alpha_{A_{12}\otimes z_{23}^*A_{23}),z_{34}^*A_{34},z_{45}^*A_{45}} & & \\
((z_{12}^*A_{12} \otimes z_{23}^*A_{23}) \otimes z_{34}^*A_{34}) \otimes z_{45}^*A_{45} & \xrightarrow[\alpha^{-1}_{z_{12}^*A_{12},z_{23}^*A_{23},z_{34}^*A_{34}}\otimes 1]{} & (z_{12}^*A_{12} \otimes (z_{23}^*A_{23} \otimes z_{34}^*A_{34})) \otimes z_{45}^*A_{45}
\end{array}$$

Here α refers via the definition (101) to the natural equivalence of the monoidal category $\mathfrak{Bun}(W)$. This way, the diagram is nothing else then the pentagon identity (MC1) for the monoidal category $\mathfrak{Bun}(W)$, evaluated on the four vector bundles $z_{i,i+1}^*A_{i,i+1}$, and thus commutes.

It remains to check axiom (2C2). The natural equivalences λ and ϱ give for every stable morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ morphisms $\lambda_{\mathcal{A}} : \mathcal{A} \circ \mathcal{L}_{\mathcal{G}} \Rightarrow \mathcal{A}$ and $\varrho_{\mathcal{A}} : \mathcal{L}_{\mathcal{G}'} \circ \mathcal{A} \Rightarrow \mathcal{A}$ of stable morphisms. In this terms, axiom (2C2) means for three bundle gerbes $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and two stable morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ the equality

$$m(\mathrm{id}_{\mathcal{A}}, \lambda_{\mathcal{A}'}) = m(\varrho_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}'}) \circ \alpha_{\mathcal{A},\mathcal{L}_{\mathcal{G}_2},\mathcal{A}'} \qquad (105)$$

of morphisms of stable morphisms from $m(\mathcal{A}, m(\mathcal{L}_{\mathcal{G}_2}, \mathcal{A}')) = (\mathcal{A}' \circ \mathcal{L}_{\mathcal{G}_2}) \circ \mathcal{A}$ to $m(\mathcal{A}, \mathcal{A}') = \mathcal{A}' \circ \mathcal{A}$. By definition, the morphism $m(\mathrm{id}_{\mathcal{A}}, \lambda_{\mathcal{A}'})$ has the covering $W = Z \times_{Y_2} Z'$ and the isomorphism

$$\beta_\lambda = (1 \otimes \lambda_{z'^*A'}) \circ (1 \otimes z'^*(y_2^* t_{\mu_2} \otimes 1)) : z^*A \otimes (z'^* y_2^* L_2 \otimes z'^*A') \to z^*A \otimes z'^*A' \quad (106)$$

of line bundles over W. The morphism $m(\varrho_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}'})$ has the same covering Z and the isomorphism

$$\beta_\varrho = (\varrho_{z^*A} \otimes 1) \circ (z^*(1 \otimes y_2^* t_{\mu_2}) \otimes z'^*A') : (z^*A \otimes z^* y_2^* L_2) \otimes z'^*A' \to z^*A \otimes z'^*A'. \quad (107)$$

Finally, since $W \cong Z \times_{Y_2} Y_2^{[2]} \times_{Y_2} Z'$, the space W is also the covering of the morphism $\alpha_{\mathcal{A}, \mathcal{L}_{\mathcal{G}_2}, \mathcal{A}'}$, which provides us with an isomorphism

$$\beta_\alpha = \alpha_{z^*A, y_2^* L_2, z'^*A'} : z^*A \otimes (z'^* y_2^* L_2 \otimes z'^*A') \to (z^*A \otimes z'^* y_2^* L_2) \otimes A' \quad (108)$$

of line bundles over W. Note that this makes only sense because W is the fibre product over Y_2, and one has $y_2 \circ z = y_2 \circ z'$. Now the equality (105) is equivalent to the equation

$$\beta_\lambda = \beta_\varrho \circ \beta_\alpha \quad (109)$$

of isomorphisms of line bundles over W. Here the two occurrences of t_{μ_2} drop out, and it remains the triangle identity (MC2)

$$1 \otimes \lambda_{z'^*A'} = (\varrho_{z^*A} \otimes 1) \circ \alpha_{z^*A, y_2^* L_2, z'^*A'} \quad (110)$$

for the monoidal category $\mathfrak{Bun}(W)$. Thus, axiom (2C2) is satisfied.

REMARK 2.4A. *A 2-category is called strict, if all the natural equivalences $\alpha_{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3}$, $\lambda_{\mathcal{A}}$ and $\varrho_{\mathcal{A}}$ are identities. From the preceding two sections it is clear that the 2-category $\mathfrak{BGrb}(M)$ is not strict.*

Note that this is an essential property of bundle gerbes. Even if one works with strictified categories of vector bundles we still obtain non-trivial natural equivalences for $\mathfrak{BGrb}(M)$, namely

$$\alpha_{\mathcal{A}_{12}, \mathcal{A}_{23}, \mathcal{A}_{34}} = \mathrm{id} \quad \text{but} \quad \lambda_{\mathcal{A}} = y^* t_\mu \otimes 1 \quad \text{and} \quad \varrho_{\mathcal{A}} = 1 \otimes y'^* t_\mu \quad (111)$$

instead of (101), (82) and (88).

2.5 1-Isomorphisms and stable Isomorphisms

In this section I relate the definition of a stable isomorphism from section 1.3 to the definition of a 1-isomorphism of the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M from section 2.1.

THEOREM 2.5A. *The 1-isomorphisms of the 2-category $\mathfrak{BGrb}(M)$ are exactly the stable isomorphisms.*

Proof . A stable morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ is a 1-isomorphism, if there is another stable morphism $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_1$ such that there are morphisms of stable morphisms – which are isomorphisms – from the composition $\mathcal{A}' \circ \mathcal{A}$ to the identity stable morphism $\mathcal{L}_{\mathcal{G}_1}$ and from the composition $\mathcal{A} \circ \mathcal{A}'$ to the identity stable morphism $\mathcal{L}_{\mathcal{G}_2}$. In this situation, the stable morphisms provide vector bundles $A \to Z$ and $A' \to Z'$. Recall that the compositions $\mathcal{A}' \circ \mathcal{A}$ and $\mathcal{A} \circ \mathcal{A}'$ provide vector bundles $\tilde{A}_1 := z^*A \otimes z'^*A'$ and $\tilde{A}_2 := z'^*A' \otimes z^*A$. The existence of morphisms $\mathcal{A}' \circ \mathcal{A} \Rightarrow \mathcal{L}_{\mathcal{G}_1}$ and $\mathcal{A} \circ \mathcal{A}' \Rightarrow \mathcal{L}_{\mathcal{G}_2}$ of stable morphisms implies

$$\mathrm{rank}(\tilde{A}_1) = \mathrm{rank}(L_1) = 1 \quad \text{and} \quad \mathrm{rank}(\tilde{A}_2) = \mathrm{rank}(L_2) = 1, \tag{112}$$

so that A and A' have to be line bundles. Hence, $\mathcal{A}$ and $\mathcal{A}'$ are stable isomorphisms.

The proof that any stable isomorphism $\mathcal{A}$ is a 1-isomorphism is given in the following by an explicit construction of an inverse stable isomorphism $\mathcal{A}^{-1}$ together with two morphisms $i_l : \mathcal{A} \circ \mathcal{A}^{-1} \Rightarrow \mathcal{L}_{\mathcal{G}_2}$ and $i_r : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathcal{L}_{\mathcal{G}_1}$ of stable morphisms. □

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two bundle gerbes over M, and let $\mathcal{A} = (A, \alpha)$ be a stable isomorphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$. Recall that $A \to Z$ is a line bundle whose curvature is fixed by axiom (SM1), and

$$\alpha \;:\; y_1^*L_1 \otimes \zeta_2^*A \;\longrightarrow\; \zeta_1^*A \otimes y_2^*L_2 \tag{113}$$

is an isomorphism of line bundles over $Z^{[2]}$ satisfying axiom (SM2).

The inverse stable isomorphism $\mathcal{A}^{-1} = (A', \alpha')$ I have to define will have the same covering Z and have the dual line bundle $A' := A^*$. Its isomorphism

$$\alpha' \;:\; y_2^*L_2 \otimes \zeta_2^*A' \;\longrightarrow\; \zeta_1^*A' \otimes y_1^*L_1 \tag{114}$$

is the following concatenation:

$$\begin{array}{c} y_2^*L_2 \otimes \zeta_2^*A' \\ \cong\big\| \\ \zeta_1^*A^* \otimes \zeta_1^*A \otimes y_2^*L_2 \otimes \zeta_2^*A^* \\ \Big\downarrow{\scriptstyle 1\otimes\alpha^{-1}\otimes 1} \\ \zeta_1^*A^* \otimes y_1^*L_1 \otimes \zeta_2^*A \otimes \zeta_2^*A^* \\ \cong\big\| \\ \zeta_1^*A' \otimes y_1^*L_1 \end{array} \tag{115}$$

Axiom (SM1) is

$$\mathrm{curv}(A') = -\mathrm{curv}(A) = -(y_2^*C_2 - y_1^*C_1) = y_1^*C_1 - y_2^*C_2 \tag{116}$$

and hence satisfied. Axiom (SM2) can easily be deduced from the fact that it is satisfied by α. So I have defined a stable isomorphism $\mathcal{A}^{-1} : \mathcal{G}_2 \to \mathcal{G}_1$.

Now I have to construct the morphisms of stable morphisms

$$i_l \;:\; \mathcal{A} \circ \mathcal{A}^{-1} \;\Longrightarrow\; \mathcal{L}_{\mathcal{G}_2} \tag{117}$$

$$i_r \;:\; \mathcal{A}^{-1} \circ \mathcal{A} \;\Longrightarrow\; \mathcal{L}_{\mathcal{G}_1}. \tag{118}$$

It suffices to construct the second one; the construction of i_l is completely analogous. Recall that the composition $\mathcal{A}^{-1} \circ \mathcal{A}$ consists of the covering $\tilde{Z} = Z \times_{Y_2} Z$, whose projections on the factors I denote by $z_i : \tilde{Z} \to Z$, the line bundle $\tilde{A} = z_1^*A \otimes z_2^*A'$ over $\tilde{Z}$, and the isomorphism $\tilde{\alpha} = (\mathrm{id} \otimes z'^*\alpha') \circ (z^*\alpha \otimes \mathrm{id})$ of line bundles over $\tilde{Z}^{[2]}$. For the covering of the morphism i_r I choose $\tilde{Z}$ itself with projections $\mathrm{id} : \tilde{Z} \to \tilde{Z}$ and the natural map

$$\tilde{y}_1 \;:\; \tilde{Z} \;\longrightarrow\; Y_1^{[2]} \;:\; (z_1, z_2) \;\longmapsto\; (y_1(z_1), y_1(z_2)). \tag{119}$$

to the covering $Y_1^{[2]}$ of the stable isomorphism $\mathcal{L}_{\mathcal{G}_1}$. This is a valid choice since the diagrams (60), here

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\mathrm{id}} & \tilde{Z} \\ {\scriptstyle \tilde{y}_1}\downarrow & & \downarrow{\scriptstyle y_1 \circ z_1} \\ Y_1^{[2]} & \xrightarrow[\pi_1]{} & Y_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{Z} & \xrightarrow{\mathrm{id}} & \tilde{Z} \\ {\scriptstyle \tilde{y}_1}\downarrow & & \downarrow{\scriptstyle y_1 \circ z_2} \\ Y_1^{[2]} & \xrightarrow[\pi_2]{} & Y_1 \end{array} \tag{120}$$

commute. Let $\tilde{y}_2$ be the canonical projection from $\tilde{Z}$ to Y_2, and let b be the embedding $b : \tilde{Z} \to Z^{[2]}$. Observe that the diagrams

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{b} & Z^{[2]} \\ & {\scriptstyle \tilde{y}_1}\searrow & \downarrow{\scriptstyle y_1} \\ & & Y_1^{[2]} \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{Z} & \xrightarrow{b} & Z^{[2]} \\ {\scriptstyle y_2}\downarrow & & \downarrow{\scriptstyle y_2} \\ Y_2 & \xrightarrow[\Delta]{} & Y_2^{[2]} \end{array} \tag{121}$$

commutes, so that one obtains an isomorphism

$$b^*\alpha \;:\; \tilde{y}_1^*L_1 \otimes z_2^*A \;\longrightarrow\; z_1^*A \otimes y_2^*\Delta^*L_2 \tag{122}$$

of line bundles over $\tilde{Z}$. Then define the isomorphism $i_r : \tilde{A} \to \tilde{y}_1^*L_1$ by the following concatenation:

$$\begin{array}{c} \tilde{A} \\ \| \cong \\ z_1^*A \otimes 1 \otimes z_2^*A^* \\ \downarrow {\scriptstyle 1 \otimes y_2^* t_{\mu_2}^{-1} \otimes 1} \\ z_1^*A \otimes y_2^*\Delta^*L_2 \otimes z_2^*A^* \\ \downarrow {\scriptstyle b^*\alpha^{-1} \otimes 1} \\ \tilde{y}_1^*L_1 \otimes z_2^*A \otimes z_2^*A^* \\ \| \cong \\ \tilde{y}_1^*L_1 \end{array} \tag{123}$$

Here I use the isomorphism $t_{\mu_2} : \Delta^*L_2 \to 1$ from Proposition 1.2b. Note that in the last step when pairing the dual bundles z_2^*A and $z_2^*A^*$ to the trivial line bundle, one encounters the condition that A is a line bundle, i.e. that $\mathcal{A}$ is a stable isomorphism.

It remains to show that i_r satisfies the axiom (MSM) for morphisms of stable morphisms. This is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} z_1^*y_1^*L_1 \otimes \tilde{\zeta}_2^*\tilde{A} & \xrightarrow{\tilde{\alpha}} & \tilde{\zeta}_1^*\tilde{A} \otimes z_2^*y_1^*L_1 \\ {\scriptstyle 1\otimes\tilde{\zeta}_2^* i_r}\downarrow & & \downarrow{\scriptstyle \tilde{\zeta}_1^* i_r \otimes 1} \\ \tilde{y}_1^*\pi_{13}^*L_1 \otimes \tilde{\zeta}_2^*\tilde{y}_1^*L_1 & \xrightarrow[\iota]{} & \tilde{\zeta}_1^*\tilde{y}_1^*L_1 \otimes \tilde{y}_1^*\pi_{24}^*L_1 \end{array} \tag{124}$$

of line bundles over $\tilde{Z}^{[2]}$. Using the definitions, this diagram is equivalent to the following one:

$$\begin{array}{ccccc} * & \xrightarrow{z_1^*\alpha\otimes 1} & * & \xleftarrow{z_2^*\alpha\otimes 1} & * \\ {\scriptstyle 1\otimes 1\otimes y_2^*\pi_2^* t_{\mu_2}}\uparrow & & & & \uparrow{\scriptstyle 1\otimes y_2^*\pi_1^* t_{\mu_2}\otimes 1} \\ * & & & & * \\ {\scriptstyle 1\otimes\tilde{\zeta}_2^*\alpha}\uparrow & & & & \uparrow{\scriptstyle 1\otimes\tilde{\zeta}_1^*\alpha} \\ * & \xleftarrow[\tilde{y}_1^*\pi_{134}^*\mu_1^{-1}\otimes 1]{} & * & \xrightarrow[\tilde{y}_1^*\pi_{124}^*\mu_1^{-1}\otimes 1]{} & * \end{array} \tag{125}$$

Observe that both arrows with t_{μ_2} commute with the arrows with $z_i^*\alpha$ and move to the top. There are two projections $\zeta_{134} : \tilde{Z}^{[2]} \to Z^{[3]}$ and $\zeta_{124} : \tilde{Z}^{[2]} \to Z^{[3]}$ regarding $\tilde{Z}^{[2]}$ as a subspace of $Z^{[4]}$. The pullback of axiom (SM2) for α along ζ_{134} gives the relation

$$(z_1^*\alpha \otimes 1) \circ (1 \otimes \tilde{\zeta}_2^*\alpha) = (1 \otimes y_2^*\Delta_{122}^*\mu_2^{-1}) \circ \zeta_{14}^*\alpha \circ (\tilde{y}_1^*\pi_{134}^*\mu_1 \otimes 1) \tag{126}$$

and its pullback along ζ_{124} gives

$$(z_2^*\alpha \otimes 1) \circ (1 \otimes \tilde{\zeta}_1^*\alpha) = (1 \otimes y_2^*\Delta_{112}^*\mu_2^{-1}) \circ \zeta_{14}^*\alpha \circ (\tilde{y}_1^*\pi_{124}^*\mu_1 \otimes 1). \tag{127}$$

Using these relations, the commutativity of the diagram reduces to the equation

$$(1 \otimes y_2^*\pi_1^*t_{\mu_2} \otimes 1) \circ (1 \otimes y_2^*\Delta_{112}^*\mu_2^{-1}) = (1 \otimes 1 \otimes y_2^*\pi_2^*t_{\mu_2}) \circ (1 \otimes y_2^*\Delta_{122}^*\mu_2^{-1}). \tag{128}$$

By Proposition 1.2b, both sides are separately the identity maps, so that the diagram commutes, and i_r is a morphism of stable morphisms. Since in every 2-category inverse 1-isomorphisms are unique up to 2-isomorphisms, for every other inverse stable isomorphism $\mathcal{A}' : \mathcal{G}' \to \mathcal{G}$ there is a morphism $\beta : \mathcal{A}' \Rightarrow \mathcal{A}^{-1}$ of stable isomorphisms to the stable isomorphism $\mathcal{A}^{-1}$ constructed here.

2.6 Holonomy of Bundle Gerbes and Bundle Gerbe Modules

Now that I defined inverses of stable isomorphisms, recall the definition 1.4a of a trivialization of a bundle gerbe as a stable isomorphism from the gerbe $\mathcal{G}$ to the canonical bundle gerbe $\mathcal{I}_\varrho$ with B-field ϱ. Concerning stable morphisms of those canonical bundle gerbes, one has the following

Proposition 2.6a. *For any stable morphism $\mathcal{A} = (A, \alpha) : \mathcal{I}_{\varrho_1} \to \mathcal{I}_{\varrho_2}$ with covering $\zeta : Z \to M$, the pair (A, α^{-1}) is an object in the descent category $\mathfrak{Des}(\mathcal{B}, \zeta)$. The vector bundle $D_\zeta^*(A, \alpha^{-1})$ over M has the curvature $\varrho_2 - \varrho_1$.*

Proof . Although this proposition can be obtained as a corollary of Theorem A.1 from the appendix, I give a direct proof. By definition, the projections $m_1 : Z \to M$ to the covering M of $\mathcal{I}_{\varrho_1}$ and $m_2 : Z \to M$ to the covering of $\mathcal{I}_{\varrho_2}$ coincide with ζ. Since the line bundle of the canonical bundle gerbes is the trivial line bundle, the isomorphism α gives an isomorphism $\alpha^{-1} : \zeta_1^* A \to \zeta_2^* A$ of line bundles over $Z^{[2]}$. Axiom (SM1) states $\mathrm{curv}(A) = \zeta^*(\varrho_2 - \varrho_1)$, and Axiom (SM2) reduces to the cocycle condition for α, so that (A, α^{-1}) is an object in the descent category $\mathfrak{Des}(\mathcal{B}, \zeta)$. □

COROLLARY 2.6B. *If $\mathcal{T}_1 : \mathcal{G} \to \mathcal{I}_{\varrho_1}$ and $\mathcal{T}_2 : \mathcal{G} \to \mathcal{I}_{\varrho_2}$ are two trivializations of the same bundle gerbe $\mathcal{G}$, the 2-form $\varrho_2 - \varrho_1$ on M is closed and has integral periods.*

Proof . By Proposition 2.5A and Corollary 1.3D the composition $\mathcal{T}_1 \circ \mathcal{T}_2^{-1} : \mathcal{I}_{\varrho_2} \to \mathcal{I}_{\varrho_1}$ is a stable isomorphism. Hence it defines a line bundle over M whose curvature is $\varrho_1 - \varrho_2$. The curvature of a line bundle is a closed 2-form with integral periods. □

This corollary is essential for the definition of the holonomy of a topological trivial bundle gerbe $\mathcal{G}$ over a two-dimensional oriented smooth manifold M. Choose a trivialization $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\varrho$ and set

$$\mathrm{hol}_{\mathcal{G}}(M) := \exp\left(2\pi \mathrm{i} \int_M \varrho\right) \in U(1). \tag{129}$$

This is independent of the choice of $\mathcal{T}$ and ϱ.

Recall further Definition 1.4B of a symmetric bundle gerbe module as a stable morphism from $\mathcal{G}$ to the canonical bundle gerbe $\mathcal{I}_\omega$. Suppose now that M is a smooth manifold with boundary, and let $\mathcal{G}$ be a bundle gerbe over M with a symmetric bundle gerbe module[5] $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$. Choose a trivialization $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\varrho$. Now, $\mathcal{E} \circ \mathcal{T}^{-1}$ is a stable morphism from $\mathcal{I}_\varrho$ to $\mathcal{I}_\omega$ defining a vector bundle E over M by Proposition 2.6A. The parallel transport of this vector bundle around ∂M gives elements in $U(\mathrm{rank}(E))$, such that their trace $\mathrm{trhol}_E(\partial M) \in \mathbb{C}$ is well-defined without respect to a fixed covering point. Then define

$$\mathrm{hol}_{\mathcal{G},\mathcal{E}}(M) := \exp\left(2\pi \mathrm{i} \int_M \varrho\right) \cdot \mathrm{trhol}_E(\partial M) \in \mathbb{C}. \tag{130}$$

This expression is well-defined: if $\mathcal{T}' : \mathcal{G} \to \mathcal{I}_{\varrho'}$ is another trivialization, by the properties of the 2-category $\mathfrak{BGrb}(M)$ there is a morphism

$$\mathcal{E} \circ \mathcal{T}'^{-1} \implies (\mathcal{E} \circ \mathcal{T}^{-1}) \circ (\mathcal{T} \circ \mathcal{T}'^{-1}) \tag{131}$$

of stable morphisms from $\mathcal{I}_{\varrho'}$ to $\mathcal{I}_\omega$. According to Proposition 2.6A one obtains the vector bundles E' and E, and a line bundle A from the stable isomorphism $\mathcal{T} \circ \mathcal{T}'^{-1} : \mathcal{I}_{\varrho'} \to \mathcal{I}_\varrho$ with curvature $\varrho - \varrho'$. The existence of the morphism (131) implies an isomorphism

[5] It would be sufficient to demand a bundle gerbe module for $\mathcal{G}$ restricted to the boundary ∂M. Since we have not defined pullbacks of bundle gerbes (and therewith restrictions), we have to assume a global bundle gerbe module here.

$$E' \cong A \otimes E, \tag{132}$$

of vector bundles over M. Then,

$$\begin{aligned} \operatorname{trhol}_{E'}(\partial M) &= \operatorname{trhol}_{A\otimes E}(\partial M) && (133)\\ &= \operatorname{hol}_A(\partial M)\cdot \operatorname{trhol}_E(\partial M) && (134)\\ &= \exp\left(2\pi \mathrm{i}\int_M \varrho - \varrho'\right)\cdot \operatorname{trhol}_E(\partial M) && (135) \end{aligned}$$

shows that the holonomy is well-defined.

Appendix: Descent stable Morphisms

In the previous publications on bundle gerbes, p.e. [MS00, Ste00, GR02], stable isomorphisms between two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over M are exactly those stable isomorphisms in the sense of Definition 1.3b, whose covering Z is the fibre product of the coverings of the bundle gerbes, $Z = Y_1 \times_M Y_2$. In this appendix I show, that there is a stable isomorphism in the sense of the previous publications between $\mathcal{G}_1$ and $\mathcal{G}_2$, if and only if there is one in the sense of Definition 1.3b. This way, the stable isomorphism classes of bundle gerbes coincide.

In fact I show more, namely that any stable morphisms with some covering Z descends to another one whose covering is the fibre product, so that both are equivalent as stable morphisms:

Theorem A.1. *For any stable morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ there is a stable morphism $\mathcal{S}_{\mathcal{A}} : \mathcal{G}_1 \to \mathcal{G}_2$ with covering $P := Y_1 \times_M Y_2$ such that $\mathcal{A}$ and $\mathcal{S}_{\mathcal{A}}$ are isomorphic objects in $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$.*

I split the proof in a sequence of constructions and propositions. The plan is the following. First I define a more suitable stable morphism $\mathcal{K} : \mathcal{G}_1 \to \mathcal{G}_2$, consisting of a vector bundle K over some space W, and of an isomorphism κ over $W^{[2]}$. This will be

$$\mathcal{K} := \mathcal{L}_{\mathcal{G}_2} \circ \mathcal{A} \circ \mathcal{L}_{\mathcal{G}_1} \tag{136}$$

and is isomorphic to $\mathcal{A}$ as objects in $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$. This stable isomorphism will have the following properties. The surjective submersion $\omega : W \to M$ factors through P,

$$\omega = p \circ w \tag{137}$$

for some map $w : W \to P$. I will define an isomorphism $k : w_1^*K \to w_2^*K$ over the two-fold fibre product $W_P^{[2]} := W \times_P W$ such that (K, k) defines an object in $\mathfrak{Des}(\mathcal{B}, w)$ and the isomorphism κ defines a morphism in $\mathfrak{Des}(\mathcal{B}, w^{[2]})$. Then I show that

$$\mathcal{S}_{\mathcal{A}} := D_w^*(\mathcal{K}) := (D_w^*(K, k), D_w^*(\kappa)) \tag{138}$$

defines a stable morphism with covering P.

Let the stable morphism $\mathcal{A}$ have covering $\zeta : Z \to M$, a vector bundle A over Z and an isomorphism α over $Z^{[2]}$. The covering $\omega : W \to M$ of the stable morphism $\mathcal{K}$ can be identified with

$$W \cong Y_1 \times_M Z \times_M Y_2 \tag{139}$$

with the obvious projections y_1, z and y_2 in the respective factors and two mixed projections,

$$y_1^2 \;:\; W \longrightarrow Y_1^{[2]} \;:\; (y_1, z, y_2) \longmapsto (y_1, y_1(z)) \tag{140}$$
$$y_2^2 \;:\; W \longrightarrow Y_2^{[2]} \;:\; (y_1, z, y_2) \longmapsto (y_2(z), y_2) \tag{141}$$

so that the vector bundle K of the stable isomorphism $\mathcal{K}$ is

$$K = (y_1^2)^* L_1 \otimes z^* A \otimes (y_2^2)^* L_2. \tag{142}$$

The isomorphism

$$\kappa : y_1^* L_1 \otimes \omega_2^* K \to \omega_1^* K \otimes y_2^* L_2 \tag{143}$$

of the stable isomorphism $\mathcal{K}$ is an isomorphism of vector bundles over $W^{[2]}$ which is given by

$$\kappa = (1 \otimes 1 \otimes (y_2^2)^* l_2) \circ (1 \otimes z^* \alpha \otimes 1) \circ ((y_1^2)^* l_1 \otimes 1 \otimes 1). \tag{144}$$

Define $w := (y_1, y_2) : W \to P$ so that equation (137) is satisfied. Now consider the two-fold fibre product of W over P, which is $W_P^{[2]} = Y_1 \times_M Z^{[2]} \times_M Y_2$. One has the relation

$$z \circ w_i = \zeta_i \circ z \tag{145}$$

as maps from $W_P^{[2]}$ to Z for $i = 1, 2$, and an embedding

$$w_M : W_P^{[2]} \to W^{[2]} : (y_1, z_1, z_2, y_2) \mapsto (y_1, z_1, y_2, y_1, z_2, y_2). \tag{146}$$

Consider the pullback of the isomorphism κ along the embedding w_M. Using the definition of the isomorphisms l_1 and l_2 from equation (76), this pullback has the form

$$w_M^* \kappa = (1 \otimes 1 \otimes (y_2^2)^* \Delta_{122}^* \mu_2^{-1}) \circ (1 \otimes k^{-1} \otimes 1) \circ ((y_1^2)^* \Delta_{112}^* \mu_1 \otimes 1 \otimes 1), \tag{147}$$

with an isomorphism $k : w_1^* K \to w_2^* K$ of line bundles over $W_P^{[2]}$ given by

$$k = ((y_1^2)^* \pi_{124}^* \mu_1 \otimes 1 \otimes 1) \circ (1 \otimes z^* \alpha^{-1} \otimes 1) \circ (1 \otimes 1 \otimes (y_2^2)^* \pi_{134}^* \mu_2^{-1}). \tag{148}$$

This composition indeed defines an isomorphism $k : w_1^* K \to w_2^* K$: the first map is an isomorphism

$$1 \otimes 1 \otimes (y_2^2)^* \pi_{134}^* \mu_2^{-1} : w_1^*(y_1^* L_1 \otimes z^* A \otimes y_2^* L_2) \to w_1^* y_1^* L_1 \otimes z^* \zeta_1^* A \otimes z^* y_2^* L_2 \otimes w_2^* y_2^* L_2$$

due to (145) and two obvious relations between $\pi_{134} \circ y_2^2$ and y_2. The second map is an isomorphism

$$1 \otimes z^* \alpha^{-1} \otimes 1 : w_1^* y_1^* L_1 \otimes z^* \zeta_1^* A \otimes z^* y_2^* L_2 \otimes w_2^* y_2^* L_2 \\ \to w_1^* y_1^* L_1 \otimes z^* y_1^* L \otimes z^* \zeta_2^* A \otimes w_2^* y_2^* L_2,$$

and the third map is an isomorphism

$$(y_1^2)^* \pi_{124}^* \mu_1 \otimes 1 \otimes 1 : w_1^* y_1^* L_1 \otimes z^{*1} y_1^* L \otimes z^* \zeta_2^* A \otimes w_2^* y_2^* L_2 \to w_2^*(y_1^* L_1 \otimes z^* A \otimes y_2^* L_2)$$

because of (145) and another relation between $\pi_{124} \circ y_1^2$ and y_1.

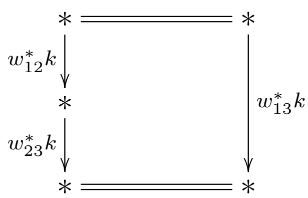
PROPOSITION A.2. *(K, k) is an object in the descent category $\mathfrak{Des}(\mathcal{B}, w)$ and κ is a morphism in the descent category $\mathfrak{Des}(\mathcal{B}, w^{[2]})$, i.e. the isomorphism $k : w_1^*K \to w_2^*K$ has the following two properties:*

(a) k satisfies the cocycle condition $w_{13}^*k = w_{23}^*k \circ w_{12}^*k$.

(b) k commutes with the isomorphism κ.

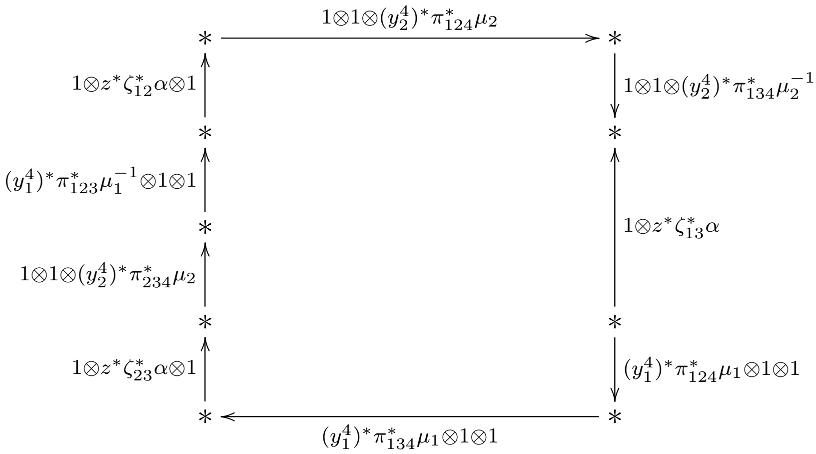
Proof .

(a) The cocycle conditions live over the space $W_P^{[3]} \cong Y_1 \times_M Z^{[3]} \times_M Y_2$ which comes with two projections $y_i^4 : W_P^{[3]} \to Y_i^{[4]}$ for $i = 1, 2$, and one projection $z : W_P^{[3]} \to Z^{[3]}$. I write the cocycle condition as a diagram



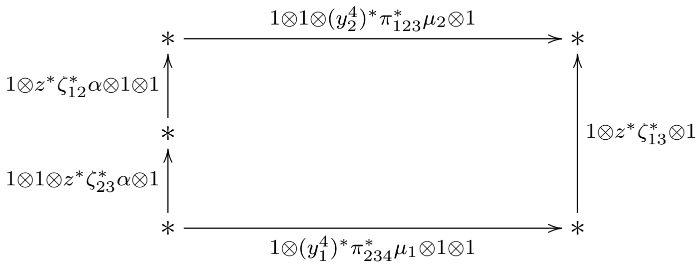
(149)

and plug in the definition (148) of k. Then I obtain



(150)

I can pull the arrow with μ_2 on the left hand side to the top and the one with μ_1^{-1} to the bottom, because they act on different factors of the tensor products than the arrows with α. Then I apply the associativity axiom (G2) of the bundle gerbe $\mathcal{G}_1$ on three arrows in the bottom, and the one for the bundle gerbe $\mathcal{G}_2$ on three arrows in the top. I end up with the diagram



(151)

which is nothing but the diagram from axiom (SM2) for α, pulled back along $z : W_P^{[3]} \to Z^{[3]}$. So it commutes.

(b) Note that the descend condition (24) for morphisms lives over $W^{[2]} \times_{P^{[2]}} W^{[2]}$, namely

$$\begin{array}{ccc} a_1^*(y_1^*L_1 \otimes \omega_2^*K) & \xrightarrow{a_1^*\kappa} & a_1^*(\omega_1^*K \otimes y_2^*L_2) \\ {\scriptstyle 1\otimes f_2^*k}\big\downarrow & & \big\downarrow{\scriptstyle f_1^*k\otimes 1} \\ a_2^*(y_1^*L_1 \otimes \omega_2^*K) & \xrightarrow[a_2^*\kappa]{} & a_2^*(\omega_1^*K \otimes y_2^*L_2) \end{array} \tag{152}$$

Here a_1 and a_2 are the projections into the two factors. Because the fibre product is over $P^{[2]}$ one obtains canonical projections $f_i = (\omega_i, \omega_i) : W^{[2]} \times_{P^{[2]}} W^{[2]} \to W_P^{[2]}$ and the relations $y_i \circ a_1 = y_i \circ a_2$ and $\omega_i \circ a_j = w_j \circ f_i$ so that the diagram is well-defined.

Again, I use the definitions of κ and k. For simplicity, I suppress the canonical projections to $Y_1^{[6]}$ and to $Y_2^{[6]}$. Then I obtain the following diagram:

$$\begin{array}{ccccccccccc}
* & \xrightarrow{\pi_{123}^*\mu_1\otimes 1\otimes 1} & * & \xrightarrow{\pi_{134}^*\mu_1^{-1}\otimes 1\otimes 1} & * & \xrightarrow{1\otimes a_1^*z^*\alpha\otimes 1} & * & \xrightarrow{1\otimes 1\otimes\pi_{126}^*\mu_2} & * & \xrightarrow{1\otimes 1\otimes\pi_{156}^*\mu_2^{-1}} & * \\
{\scriptstyle 1\otimes 1\otimes 1\otimes\otimes\pi_{246}^*\mu_2^{-1}}\big\downarrow & & & & & & & & & & \big\downarrow{\scriptstyle 1\otimes 1\otimes\pi_{135}^*\mu_2^{-1}\otimes 1} \\
* & & & & & & & & & & * \\
{\scriptstyle 1\otimes 1\otimes f_2^*z^*\alpha^{-1}\otimes 1}\big\downarrow & & & & & & & & & & \big\downarrow{\scriptstyle 1\otimes f_1^*z^*\alpha^{-1}\otimes 1\otimes 1} \\
* & & & & & & & & & & * \\
{\scriptstyle 1\otimes\pi_{236}^*\mu_1\otimes 1\otimes 1}\big\downarrow & & & & & & & & & & \big\downarrow{\scriptstyle \pi_{135}^*\mu_1\otimes 1\otimes 1\otimes 1} \\
* & \xrightarrow[\pi_{126}^*\mu_1\otimes 1\otimes 1]{} & * & \xrightarrow[\pi_{156}^*\mu_1^{-1}\otimes 1\otimes 1]{} & * & \xrightarrow[1\otimes a_2^*z^*\alpha\otimes 1]{} & * & \xrightarrow[1\otimes 1\otimes\pi_{146}^*\mu_2]{} & * & \xrightarrow[1\otimes 1\otimes\pi_{356}^*\mu_2^{-1}]{} & *
\end{array}$$

I permute the order of isomorphisms where it is possible since they act on different factors of tensor products, in the way that I get the diagram:

$$\begin{array}{ccccccccccc}
* & \xrightarrow{\pi_{134}^*\mu_1^{-1}\otimes 1\otimes 1} & * & \xrightarrow{1\otimes 1\otimes f_2^*z^*\alpha\otimes 1} & * & \xrightarrow{1\otimes a_1^*z^*\alpha\otimes 1} & * & \xrightarrow{1\otimes 1\otimes 1\otimes\otimes\pi_{246}^*\mu_2^{-1}} & * & \xrightarrow{1\otimes 1\otimes\pi_{126}^*\mu_2} & * \\
{\scriptstyle \pi_{123}^*\mu_1^{-1}\otimes 1\otimes 1}\big\downarrow & & & & & & & & & & \big\downarrow{\scriptstyle 1\otimes 1\otimes\pi_{156}^*\mu_2^{-1}} \\
* & & & & & & & & & & * \\
{\scriptstyle 1\otimes\pi_{236}^*\mu_1\otimes 1\otimes 1}\big\downarrow & & & & & & & & & & \big\downarrow{\scriptstyle 1\otimes 1\otimes\pi_{135}^*\mu_2^{-1}\otimes 1} \\
* & & & & & & & & & & * \\
{\scriptstyle \pi_{126}^*\mu_1\otimes 1\otimes 1}\big\downarrow & & & & & & & & & & \big\downarrow{\scriptstyle 1\otimes 1\otimes\pi_{356}^*\mu_2} \\
* & \xrightarrow[\pi_{156}^*\mu_1^{-1}\otimes 1\otimes 1]{} & * & \xrightarrow[\pi_{135}^*\mu_1^{-1}\otimes 1\otimes 1\otimes 1]{} & * & \xrightarrow[1\otimes a_2^*z^*\alpha\otimes 1]{} & * & \xrightarrow[1\otimes f_1^*z^*\alpha\otimes 1\otimes 1]{} & * & \xrightarrow[1\otimes 1\otimes\pi_{146}^*\mu_2]{} & *
\end{array}$$

Due to the relations

$$z \circ f_2 = \zeta_{24} \circ z \quad \text{and} \quad z \circ a_1 = \zeta_{12} \circ z \tag{153}$$

and

$$z \circ f_1 = \zeta_{13} \circ z \quad \text{and} \quad z \circ a_2 = \zeta_{34} \circ z \tag{154}$$

one is able to apply the axiom (SM1) for the isomorphism α two times, namely

$$(a_1^*z^*\alpha \otimes 1) \circ (1 \otimes f_2^*z^*\alpha) = (1 \otimes \pi_{124}^*\mu_2^{-1}) \circ z^*\zeta_{14}^*\alpha \circ (\pi_{346}^*\mu_1 \otimes 1) \quad (155)$$

$$(f_1^*z^*\alpha \otimes 1) \circ (1 \otimes a_2^*z^*\alpha) = (1 \otimes \pi_{134}^*\mu_2^{-1}) \circ z^*\zeta_{14}^*\alpha \circ (\pi_{356}^*\mu_1 \otimes 1) \quad (156)$$

After this replacement, all occurrences of the isomorphisms μ_1 and μ_2 disappear by using axiom (G2) for both ones three times each. One ends up with the equality $\zeta_{14}^*\alpha = \zeta_{14}^*\alpha$. So the diagram commutes.

□

Now the proof of Theorem A.1 is completed with the following proposition.

PROPOSITION A.3. *Let $\mathcal{K} = (K, \kappa) : \mathcal{G}_1 \to \mathcal{G}_2$ be a stable isomorphism with covering $\omega : W \to M$, let $p : P \to M$ be another surjective submersion with a map $w : W \to P$ such that $\omega = p \circ w$ and let $k : w_1^*K \to w_2^*K$ be an isomorphism of vector bundles over $W_P^{[2]}$ such that (K, k) is an object and κ is a morphism in the descend category $\mathfrak{Des}(\mathcal{B}, w)$. Then,*

(a) $D_w^*(\mathcal{K}) := (D_w^*(K, k), D_w^*(\kappa))$ is a stable isomorphism with covering P.

(b) $D_w^*(\mathcal{K})$ and $\mathcal{K}$ are isomorphic objects in $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$.

Proof . (a) The axioms for $D_w^*(\mathcal{K})$ and $\mathcal{K}$ are equivalent because $\mathcal{B}$ is a stack and hence $\mathcal{D}_w^*$ is an equivalence of categories. (b) It is easy to construct a morphism $\beta : D_w^*(\mathcal{K}) \Rightarrow \mathcal{K}$ by choosing the covering W with maps $\mathrm{id} : W \to W$ to the covering of $\mathcal{K}$ and $w : W \to P$ to the covering of $D_w^*(\mathcal{K})$. Then the stack $\mathcal{B}$ provides an isomorphism $\beta : w^*D_w^*(K, k) \to K$ satisfying the axiom (MSM). □

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